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D–branes, Rolling Tachyons and Vacuum String Field Theory

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Abstract

In this thesis we discuss classical solutions and open string vacua of Open String Field Theory.

After a brief introduction which contains the basic ideas and motivations for this research, we give an outline of the structure of Open String Field Theory and its relation with Boundary Conformal Field Theories.

We then concentrate on the $*$–product, that makes the string Hilbert space a non–commutative algebra. We review the construction of the three string vertex for the matter and the ghost sector, for the latter case we provide a new formulation which allows to treat ghost zero modes on the same footing as non zero modes, thus providing a more compact and economical structure. All the Neumann coefficients are then diagonalized and the corresponding eigenvalues and eigenvectors are derived. This preliminary technical part is necessary for the explicit computations that are presented in the rest of the material of this thesis.

We turn our discussion to the structure of Open String Field Theory around the tachyon vacuum and we define the Vacuum String Field Theory model. We show how our formulation of the ghost $*$–product allows for a simple derivation of the universal ghost solution, common to all the classical solutions of VSFT that represent D–branes systems of any kind.

We then proceed with the construction of a solution in the matter sector (the Dressed Sliver) which represents a D25–brane background, as seen from the tachyon vacuum. We derive a deformation technique (dressing) which allows for an unambiguous definition of the string coupling constant as an emergent quantity from the regularization procedure. We show how this formalism can be extended to derive solutions representing multiple D25–branes and lower dimensional D–branes (lump solutions).

The next topic we address is the systematic study of the linearized equations of motion around the Dressed Sliver background. We show, up to level 3, that all the open string spectrum arises with the correct Virasoro constraints: this is possible thanks to the dressing deformation (as far as the transversality condition for the $U(1)$ gauge field is concerned) and due to a regularization of the midpoint degrees of freedom which allows for a proper definition of the massive modes. We indeed show in detail that all the physical excitations arise from the midpoint and that our regularization allows to consistently deal with the singular midpoint structure of VSFT.

We further extend our analysis of the perturbative spectrum to systems of N parallel Dp–branes, showing how Chan Paton factors are automatically generated from the equation of motions and how they are related to the left/right splitting of VSFT classical solutions. We then derive the open string spectrum on the Higgs phase given by N parallel separated Dp–branes: the shift in the mass formula for strings stretched between different branes emerges from a breakdown of associativity at the midpoint degree of freedom. We evaluate this anomaly by means of wedge–states regularization, obtaining perfect agreement with the known result. We elaborate on the fact that the dynamical change in the boundary condition, from Neumann to Dirichlet, is again encoded in midpoint subtleties and that stretched states undergo a consistent change in boundary conditions between the left/right parts of the string.

The remaining part of the presented material is devoted to the derivation of time–dependent solutions which represent the rolling tachyon BCFT in the VSFT framework. This new kind of solutions are obtained from Wick rotation of codimension 1 lump solutions: however we show that we have to use an unconventional lump solution with a Neumann coefficient which is inverted w.r.t. the conventional case in the discrete spectrum. We generalize our rolling tachyon solution to the case of an E–field background in both tangential and transverse directions and we finish our presentation with a solution representing macroscopic fundamental strings charged by the background E–field.

We conclude this dissertation with a number of important unsolved questions that, in our opinion, merit further effort.

The material we are presenting is extracted from the publications [1, 2, 3, 4, 5, 6, 7, 8].

Contents

Introduction

String Theory is, at the time of writing, a very (if not the only) promising way to describe our universe in a consistent and unified theoretical framework. It provides a perturbative formulation of quantum gravity and it incorporates non abelian gauge theories. Our perturbative understanding of string theory is well established and leads to the formulation of the celebrated 5 (super)string models (Type IIA/B, $SO(32)$–Type I, $SO(32)/E_8 \times E_8$–Heterotic). These theories are (perturbatively) theories of open (type I) and closed (the others) supersymmetric strings in ten dimensional flat space time; such strings vibrate generating (as harmonics) an infinite tower of particles, some of them are massless with appropriate polarization tensors. The effective field theory of such massless fields is a supergravity theory in ten dimension (coupled to Super Yang Mills in the Type I and Heterotic cases).

In these supergravity theories there are black–hole like solutions which are extended in space. These solutions have a definite tension (mass per unit volume) and are charged by the massless p–forms of the corresponding string spectrum / supergravity multiplet. One of the main results of the last decade is the recognition that such supergravity solitons admit a microscopic string theory description: they are Dirichlet branes (D–branes). They can be described as hypersurfaces in space time on which open strings are constrained to end (in this sense $SO(32)$–type I is a theory of (unoriented) open strings ending on 32 space–time filling D9–branes). However these objects are not just boundary conditions, they are genuine dynamical objects: they are physical sources for closed strings. This can be understood in the following way. Imagine to have 2 parallel D–branes and consider an open string connecting the two D–branes, then consider the one loop partition function of this string; graphically this corresponds to a cylinder connecting the two D–branes which, in turn, can be interpreted as an exchange of a closed string between two sources.

This example shows that open and closed strings are deeply intertwined and cannot be studied separately: a theory of open strings generates closed string poles at one loop and, on the other hand, closed strings are sourced by the D–branes on which open strings live on.

The discovery of D–branes has been a key element to understand that the five distinct string theories just mentioned above (plus a still not defined theory, dubbed M–Theory, whose low energy limit is eleven dimensional supergravity) are related to each others by suitable duality transformations. This web of dualities points towards the existence of a single theory to be formulated with a unique and complete set of variables, which reduces to the known superstring theories on particular points of its moduli space. This still hypothetical theory should give a non–perturbative definition of quantum gravity and, as such, should be background independent: space time itself should arise dynamically as a coherent state from nothing.

In this scenario the relevance of tachyons is fundamental. From a point particle point of view tachyons are particles that propagates faster than light, violating causality. Equivalently they are relativistic particles with negative $mass^2$. This is however a fake understanding of tachyons and, from a first quantized point of view, a tachyon is just an inconsistency of a theory. On the other hand, in field theory, we know that the concept of mass (of a scalar) just arises from the quadratic term of the scalar potential around a stationary point. If the quadratic term is positive then, by quantizing the theory, we get a massive particle but, if the quadratic term is negative, we get a tachyon that simply indicates that we are quantizing the theory on an unstable vacuum, perturbation theory breaks down and some phase transition takes place, driving the theory to a new stable vacuum, with a different perturbative spectrum.

There are many unstable vacua in string theory, signaled by corresponding tachyons on the string spectrum. The simplest example to think about is just the 26 dimensional closed bosonic string in flat space. This theory is not supersymmetric, does not contain fermions (it is in fact not very realistic...) and has a low lying state that is a tachyon. Although this is the simplest tachyon that one encounters in the first study of string theory, it is also the most mysterious one: indeed it signals the instability of the 26 dimensional bosonic spacetime itself and it is not clear at all if some decaying process can bring it to a new stable spacetime (maybe the 10 dimensional supersymmetric spacetimes?).

There is another very simple tachyon, the open string one. A theory of open strings is however a theory of D–branes, as open strings are just excitations of them. If an open string theory contains a tachyon, this can only mean that the corresponding D–brane system is unstable. In particular all bosonic D–branes are not charged and they all contain a tachyon in their spectra. In this sense the 26 dimensional open string tachyon is just the signal of the instability of the space filling D25–brane who does not have any charge protecting it. These examples might look academical as they are in the realm of the bosonic string, there are however other open string tachyons in the supersymmetric string, the bosonic case is just a simpler example of the same kind of phenomenon.

Type–IIA/B theories contain stable D–branes of even/odd dimensions, these branes are stable because they carry RR–charge and source the corresponding RR–massless fields of the closed string sector, they are BPS states and break half of the supersymmetry of the bulk space time. From an open string point of view they don't contain tachyons in their spectra because the NS tachyon has been swept away by the GSO projections needed to keep modular invariance at one loop. Since these branes are charged they possess an orientation given by the corresponding RR–p form which is a volume form for the brane's worldvolume. A D–brane of opposite orientation is just a D–brane with opposite RR–charge, an anti–D–brane. Now, if a D–brane and an anti–D–brane are placed parallely at a distance less then the fundamental string length, there is a tachyon corresponding to the lowest state of open strings stretched between the two branes, that arises because such open strings undergo the opposite GSO projection. This tachyon is just the signal that a D–brane/anti–D–brane system is unstable as it does not posses a global RR–charge.

There are also single D–branes which are unstable, these are the branes of wrong dimensionality (odd for Type–IIA, even for Type–IIB), the non–BPS D–branes. The instability is due to the fact that these branes are not charged as there are no RR–p forms in the closed string sector that can couple to them. And there is again the corresponding tachyon in the open string sector coming from the opposite GSO–projection w.r.t. the stable case.

Where these instabilities drive the theory? Is there a stable vacuum to decay to? The answer (at least for the case of open string tachyons) is yes: an unstable system of branes decays to a vacuum where it ceases to exist and its mass is converted in closed string radiation that can propagate in the bulk. This phenomenon is known as Tachyon Condensation.

Why the study of tachyon condensation is important? It is so because the decay of unstable objects is a physical process that interpolates between two different vacua (the unstable one and the stable one). In other words, tachyon condensation is a natural path to explore the (open)–string landscape and, hence, to address in an explicit physical example the study of the elusive concept of background independence.

Needless to say that open string tachyon condensation is just (one of) the starting point(s) for the project of a background–independent formulation of string theory. The mysterious closed string tachyon (who represents the instabilities of space–time itself) still waits for a convincing interpretation. Nevertheless it appears that the physics of open tachyon condensation is still rich enough to get insights into non–perturbative string theory. This is so because of the profound (an not yet fully understood) relation between open and closed strings.

One of the latest biggest achievements of string theory is the AdS/CFT correspondence which states (in its strongest formulation) that quantum Type–IIB closed string theory on $AdS_5 \times S^5$ with N units of RR 5–form flux is dual to $\mathcal{N} = 4$ $U(N) - SYM$ theory which lives on the projective 4–dimensional boundary of AdS. This correspondence basically states that a (perturbative) quantum theory of gravity on a given background is fully captured by a Yang–Mills theory which is, in turn, the low energy limit of open string theory on N D3–branes in flat space: in other words the open strings dynamics on D–branes in flat space gives us a quantum theory of gravity in a space time which is the result of the back–reaction of the branes on the original flat geometry. This is not the stating that the full closed string Hilbert space (with all the possible changes in the closed string background) is captured by a particular D–branes configuration, but it means that a complete quantum formulation of open string theory on such D–brane configuration gives a consistent and unitary quantum theory of gravity on a given spacetime. A change in the D–brane system produces a different back reaction, hence a different spacetime. It is maybe too much optimistic to think that all closed strings background can be obtained in this way, but this is certainly an interesting way of thinking at background independence.

These examples show that, even if we only know the perturbative expansion around some particular background, there are quite convincing physical reasons to believe that we can understand how string theory backgrounds are dynamically connected. However we have to face the problem that the perturbative formulation of string theory is explicitly non background independent. This in fact is mostly a consequence of the first quantized formulation. History teaches us that the most complete theory of particles has been achieved by passing from first quantization to second quantization, that is from Quantum Mechanics to Quantum Field Theory. It is only in the framework of QFT that one can have control of the vacua of a theory and how such vacua are dynamically connected via nonperturbative effects (tunneling, dynamical symmetry breaking, confinement, etc...). In the theory of particles we see that the right language to describe physics is to promote every particle with a corresponding space–time field and then proceeding with quantization. What about strings? Even in first quantization we see that the quantum fluctuations of a single string give rise to an infinite set of particles: some of them are massless, some of them may be tachyonic and infinite of them are massive. Passing from first quantization to second quantization leads to a QFT with an infinite number of space time fields: the task seems impossible both from a conceptual (infinity means no knowledge in physics) and from a computational (infinite interactions for a given physical process) point of view. However string theory is not just a theory of infinite interacting particles, there is order inside. What marks the difference with respect to particles is the conformal symmetry of the worldsheet theory: this symmetry gives us a consistent and unique interacting scheme (at every order in perturbation theory) starting with non interacting strings. This is like having a rule that gives us (unambiguously) vertices of Feynmann diagrams from the free propagators! In this sense first quantized string theory contains informations about the full non perturbative theory, they are just hidden inside.

A vacuum of string theory is identified once the string propagates in such a way that the corresponding worldsheet theory is conformal, in other words a vacuum of string theory is a two dimensional conformal field theory. What about the string spectrum (the infinite on–shell particles obtained from the vibration of the string)? They are perturbations of the vacuum, hence they correspond to (infinitesimal) deformations of the underlying conformal field theory. However these are not generic deformations but are such as to preserve conformal symmetry, they are marginal deformations. In this language the string's landscape has an intriguing description: it is the space of two–dimensional field theories. Some points in this space are conformal field theories and correspond to exact string backgrounds (vacua). Around each vacuum there are marginal directions which deform the CFT while maintaining conformal invariance, this infinitesimal deformations are the string excitations around that particular vacuum. Some of these infinitesimal deformations can be exponentiated to a finite one, giving a one parameter family of CFT's/strings vacua. There can be also vacuum points which cannot be connected through (time independent) marginal deformations but that are the result of an RG–flow to some IR fixed point. In this language non–perturbative string theory can be identified with the dynamics of two dimensional field theories. It is evident that such an understanding is equivalent (at least classically) to a second quantized formulation of string theory: a String Field Theory.

In a String Field Theory framework, the basic degrees of freedom are all the possible deformations (string fields) of a given reference conformal field theory one starts with. Such theories admit classical solutions which are in one–to–one correspondence to exact backgrounds of string theory which, in principle, can be completely disconnected from the starting background. They also have, being "field" theories, an off–shell extension of the corresponding first quantized theory: hence they can properly describe non perturbative transitions between different vacua. There are formulations of closed and open string field theories.

While closed string field theory has a complicated non polynomial form that has proven to be resistent to any kind of analytic treatment, Open String Field Theory has a remarkable simple structure which is of Chern–Simons form. The theory is simple enough to do numerical studies on the structure of its vacua. It is fair to say that a complete formulation (where explicit computations can be performed) exists up to now only for the bosonic open string and for the NS sector of the open superstring. In both cases a study of tachyon condensation has been proved possible and a non trivial tachyon potential has been seen to emerge from the level truncated action of (Super) Open String Field Theory.

But before to enter in the review of the discovery of these new non perturbative vacua, it is worth to give a concrete definition of Open String Field Theory (chapter 1) and a detailed study of the cubic interaction term that allows for the non trivial dynamics of tachyon condensation (chapter 2). We will take this discussion again in chapter 3

Chapter 1

Open String Field Theory: an outline

Open String Field Theory, [9], is a second quantized formulation of the open bosonic string. Its fundamental degrees of freedom are the open string fields, namely all kinds of vertex operators (primary and not primary) that can be inserted at the boundary of a given bulk CFT, which represents a (once and for all) fixed closed string background, for example flat space–time.

The explicit action of OSFT is derived starting from the perturbative vacuum representing a given (exactly solvable) Boundary Conformal Field Theory. In most application this $BCFT$ is the D25–brane's one, with Neumann boundary conditions on all the (non–interacting) space–time directions.

In this chapter we'll be rather formal and will concentrate on the abstract properties of the various objects that define the string field theory action, we will give precise and computable definitions starting from the next chapter.

1.1 The kinetic action

The kinetic part of the action defines the on–shell states on a given open string background. In the case of a D25–brane it takes the form

$$S_{kin}[\psi] = \langle \psi, Q_{BRST}\psi \rangle \tag{1.1}$$

In the above formula ψ is a classical string field: a generic vertex operator of ghost number 1; Q_{BRST} is the first quantized BRST operator and the inner product $\langle \cdot, \cdot \rangle$ is the *bpz* inner product, relative to the $BCFT_0$ in consideration (the D25–brane); namely

$$\langle \phi, \psi \rangle \quad = \quad \langle I \circ \phi(0)\ \psi(0) \rangle_{BCFT_0} \tag{1.2}$$

$$I(z) = -\frac{1}{z}$$

By varying the kinetic action we get the linearized equation of motion

$$Q_{BRST}|\psi\rangle = 0 \tag{1.3}$$

which is the usual on–shell condition for vertex operator of ghost number 1. The action possesses a (reducible) gauge invariance

$$\delta\psi = Q_{BRST}|\Lambda\rangle \tag{1.4}$$

for a generic string field Λ of ghost number 0, this gauge symmetry is reducible because we can have string fields of any negative ghost number, hence we have to mod out the previous gauge transformation by Q_{BRST}–closed string fields of ghost number zero, and so on.

The critical dimension $D = 26$ is obtained by the nilpotency condition of the BRST operator

$$Q^2_{BRST} = 0 \quad \Leftrightarrow \quad D = 26 \tag{1.5}$$

We see that the non trivial solution of the linearized equation of motion are in one to one correspondence with the usual open string spectrum and, more generally, with the (infinitesimal) marginal deformations of $BCFT_0$.

1.2 The interacting action

The kinetic action describes the small fluctuations of the perturbative vacuum that are identified by the (infinitesimal) boundary marginal deformations of $BCFT_0$. The simplest covariant way to introduce interactions is to add a cubic term to the action

$$S[\psi] = -\frac{1}{g_o^2}\left(\frac{1}{2}\langle\psi, Q_{BRST}\psi\rangle + \frac{1}{3}\langle\psi, \psi * \psi\rangle\right) \tag{1.6}$$

Note that we have normalized the action with the open string coupling constant g_o.

The cubic term is constructed using the operation $*$ which is an associative non–commutative product in the Hilbert space of string fields.

$$(\psi_1 * \psi_2) * \psi_3 = \psi_1 * (\psi_2 * \psi_3) \tag{1.7}$$

The Q_{BRST} operator is a derivation of the $*$–algebra

$$Q_{BRST}(\psi_1 * \psi_2) = (Q_{BRST}\psi_1) * \psi_2 + (-1)^{|\psi_1|}\psi_1 * (Q_{BRST}\psi_2), \tag{1.8}$$

where $|\psi_1|$ is the grassmannality of the string field, the ghost number in the case of the bosonic string.

Using the fact that

$$\langle Q_{BRST}(...)\rangle_{BCFT_0} = 0 \tag{1.9}$$

one can easily prove that the above action is invariant under the following gauge transformation

$$\delta\psi = Q_{BRST}|\Lambda\rangle + [\Lambda, \psi]_* \tag{1.10}$$

This infinitesimal gauge transformation can be extended to a finite one

$$\psi' = \mathrm{e}^{\Lambda}\,(Q_{BRST} + \psi)\,\mathrm{e}^{-\Lambda}, \tag{1.11}$$

where the exponentials are in the $*$–product sense. The addition of just a cubic coupling to make the action interacting can seem a bit arbitrary and even to much simple. This is however the only consistent choice one can make as it gives a unique and complete covering of the moduli space of Riemann surfaces with boundary, [112]. In other words, any worldsheet of an arbitrary number of external legs and loops can be uniquely recovered by an appropriate Feynmann diagram build with the cubic vertex and the propagator.

The equation of motion are obtained by varying the action with respect to ψ and reads

$$Q_{BRST}|\psi\rangle + |\psi\rangle * |\psi\rangle = 0 \tag{1.12}$$

Given a solution ψ_0 of the equation of motion one can shift the the string field in the following way

$$\psi = \psi_0 + \phi \tag{1.13}$$

Then the action can be rewritten as

$$S[\psi] = S[\psi_0] + -\frac{1}{g_o^2}\left(\frac{1}{2}\langle\phi, Q_{\psi_0}\phi\rangle + \frac{1}{3}\langle\phi, \phi * \phi\rangle\right) \tag{1.14}$$

where the new kinetic operator Q_{ψ_0} is defined

$$Q_{\psi_0}\phi = Q_{BRST}\phi + \{\psi_0, \phi\}_* \tag{1.15}$$

The quantity $S[\psi_0]$ is the action evaluated at the classical solution, if this solution is static (it has no kinetic energy), then this quantity corresponds to the static energy of ψ_0, in particular

$$-\frac{S[\psi_0]}{V^{(26)}} = \tau_{\psi_0} \tag{1.16}$$

where τ_{ψ_0} is the *tension* of ψ_0, the space–averaged energy mod space.

Although the equation of motion can be written in a very simple way, it is still a challenge to find exact analytic solutions for it. Why it is so difficult? We will encounter in the next chapter the explicit definition of the star product, but we can anticipate that such an interaction couples all the modes of the strings, so the corresponding equations of motion for space–time fields are completely entangled between them. One can however proceed numerically with a truncation of the string spectrum. Instead of working with the infinite number of fields contained in the off–shell string field one can just stop at a certain value of the N (level operator) eigenvalue. This procedure, called level truncation, has proven to be convergent: as the level is increased the results converge to some finite and well defined limit.

While the level truncation technique is very useful to study numerically the tachyon potential, it is nevertheless an approximation scheme that hides the analytic properties of the tachyon vacuum. There is also another reason (more connected to the very structure of the SFT action) of why the equations of motion are so hard to be explicitly solved. All the operations that makes the theory interacting (the $*$ product and the *bpz* inner product) are factorized in the matter and ghost degrees of freedom. This is so because they corresponds to evaluating certain BCFT correlators on the disk, and such correlators obviously factorizes in the matter and ghost degrees of freedom. In particular the cubic term in the action does not mix matter and ghosts. However the kinetic term does. This is due to the fact the the BRST operator is *not* matter–ghost factorized. The reason why it is not is evident: it is like this because it has to reproduce the BRST quantization of the string on the D25–brane and in such a procedure matter and ghost degrees of freedom are necessarily coupled. Can we still get a physical theory by completely disentangle the two sector? The answer is yes: string field theory at the tachyon vacuum has a singular representation in which the BRST operator can be taken to be pure ghost. The main topic of this thesis is to analyze in detail all the physics that we can extract by starting from a SFT action whose kinetic term does not mix matter and ghost degrees of freedom. We will see that, although the perturbative spectrum given by the kinetic term is completely trivial (it represents the open string excitations around the tachyon vacuum, which are absent), still there is a rich non perturbative structure that allows to reproduce in an exact analytic way all the single and multiple D–brane systems with their correct open string spectra around them. But before to enter in such a topic a detailed study of the $*$–product via the three string vertex is in order.

Chapter 2

The $*$–product

The key element that makes String Field Theory an interacting theory is the promotion of the string field Hilbert space to a non commutative algebra. As already said in the previous chapter this is achieved by introducing a multiplication rule between string fields, the $*$ product. It is time now to explore its definition and its properties. We will first give an heuristic definition based only on the embedding coordinates in the target space $X^\mu(\sigma)$ (the *matter* sector). The matter string field can be understood as a functional of the string embedding coordinates (Schrodinger representation)

$$|\psi\rangle \quad \Rightarrow \quad \psi[X^\mu(\sigma)] = \langle X^\mu(\sigma)|\psi\rangle \tag{2.1}$$

$$\tag{2.2}$$

the states $|X^\mu(\sigma)\rangle$ are the open string position eigenstates

$$\begin{aligned} \hat{X}^\mu(\sigma)\,|X^\mu(\sigma)\rangle &= X^\mu(\sigma)\,|X^\mu(\sigma)\rangle && (2.3)\\ \langle X^\mu(\sigma)|X^\nu(\sigma')\rangle &= \eta^{\mu\nu}\delta(\sigma-\sigma') && (2.4)\end{aligned}$$

The worldsheet parameter σ spans the whole open string and it lies in the interval $[0,\pi]$. For the definition of the $*$ product it is necessary to split the string into its left and right part, so we define

$$\begin{aligned} \hat{l}^\mu(\sigma) &= \hat{X}^\mu(\sigma) \qquad && 0 \le \sigma < \frac{\pi}{2} && (2.5)\\ \hat{r}^\mu(\sigma) &= \hat{X}^\mu(\pi-\sigma) \qquad && \frac{\pi}{2} < \sigma \le \pi && (2.6)\end{aligned}$$

The midpoint $\sigma = \frac{\pi}{2}$ cannot be left/right decomposed so we will treat it as a separate coordinate (even if it is a part of a continuum).

$$\hat{x}^\mu_m = \hat{X}^\mu\left(\frac{\pi}{2}\right) \tag{2.7}$$

given these definitions the string field can be expressed as a functional of the midpoint and left/right degrees of freedom

$$\psi[X^\mu(\sigma)] = \psi[x^\mu_m;\, l^\mu(\sigma), r^\mu(\sigma)] \tag{2.8}$$

The bpz inner product can be expressed as a functional integration with respect *all* degrees of freedom

$$
\begin{aligned}
\langle\psi|\phi\rangle &= \int \mathcal{D}X(\sigma)\,\langle\psi|X(\sigma)\rangle\,\langle X(\sigma)|\phi\rangle \\
&= \int \mathcal{D}X(\sigma)\,\psi[X(\pi-\sigma)]\,\phi[\sigma] \\
&= \int dx_m \mathcal{D}l(\sigma)\mathcal{D}r(\sigma)\,\psi[x_m;\,l(\sigma),r(\sigma)]\,\phi[x_m,r(\sigma),l(\sigma)] \qquad (2.9)
\end{aligned}
$$

Note that this operation consists in gluing two strings with opposite left/right orientation. Since all the degrees of freedom are integrated, one is left with just a pure number. This is reminiscent of the trace of the product of two infinite matrices.

The star product between two string fields is another string field defined in the following way

$$
(\psi * \phi)[x_m;\,l(\sigma),r(\sigma)] = \int \mathcal{D}y(\sigma)\psi[x_m;\,l(\sigma),y(\sigma)]\phi[x_m;\,y(\sigma),r(\sigma)] \qquad (2.10)
$$

This operation consists in identifying the left half of the first string with the right half of the second string, integrating the overlapping degrees of freedom as to reproduce a third string. This is analogous to the multiplication of two infinite matrices.

A very convenient representation for explicit computations of this operation is via the definition of the 3–strings vertex. This object lives on three copies of the string Hilbert space and defines the $*$ product in the following way

$$
{}_3\langle\psi * \phi| = {}_{123}\langle V_3|\,|\psi\rangle_1|\phi\rangle_2 \qquad (2.11)
$$

The following sections are devoted to a detailed study of the three string vertex in the matter and in the ghost sector.

2.1 Three strings vertex and matter Neumann coefficients

The three strings vertex [9, 52, 53] of Open String Field Theory is given by

$$
|V_3\rangle = \int d^{26}p_{(1)}d^{26}p_{(2)}d^{26}p_{(3)}\delta^{26}(p_{(1)}+p_{(2)}+p_{(3)})\exp(-E)\,|0,p\rangle_{123} \qquad (2.12)
$$

where

$$
E = \sum_{a,b=1}^{3}\left(\frac{1}{2}\sum_{m,n\geq 1}\eta_{\mu\nu}a_m^{(a)\mu\dagger}V_{mn}^{ab}a_n^{(b)\nu\dagger} + \sum_{n\geq 1}\eta_{\mu\nu}p_{(a)}^{\mu}V_{0n}^{ab}a_n^{(b)\nu\dagger} + \frac{1}{2}\eta_{\mu\nu}p_{(a)}^{\mu}V_{00}^{ab}p_{(b)}^{\nu}\right) \qquad (2.13)
$$

Summation over the Lorentz indices $\mu,\nu = 0,\ldots,25$ is understood and η denotes the flat Lorentz metric. The operators $a_m^{(a)\mu}, a_m^{(a)\mu\dagger}$ denote the non–zero modes matter oscillators of the a–th string, which satisfy

$$
[a_m^{(a)\mu}, a_n^{(b)\nu\dagger}] = \eta^{\mu\nu}\delta_{mn}\delta^{ab}, \qquad m,n\geq 1 \qquad (2.14)
$$

$p_{(r)}$ is the momentum of the a–th string and $|0,p\rangle_{123} \equiv |p_{(1)}\rangle \otimes |p_{(2)}\rangle \otimes |p_{(3)}\rangle$ is the tensor product of the Fock vacuum states relative to the three strings. $|p_{(a)}\rangle$ is annihilated by the annihilation operators $a_m^{(a)\mu}$ and it is eigenstate of the momentum operator $\hat{p}^\mu_{(a)}$ with eigenvalue $p^\mu_{(a)}$. The normalization is

$$\langle p_{(a)}|\, p'_{(b)}\rangle = \delta_{ab}\delta^{26}(p+p') \tag{2.15}$$

The symbols $V^{ab}_{nm}, V^{ab}_{0m}, V^{ab}_{00}$ will denote the coefficients computed in [52, 53]. We will use them in the notation of Appendix A and B of [54] and refer to them as the *standard* ones. The notation V^{rs}_{MN} for them will also be used at times (with $M(N)$ denoting the couple $\{0,m\}$ $(\{0,n\})$.

An important ingredient in the following are the *bpz* transformation properties of the oscillators

$$bpz(a_n^{(a)\mu}) = (-1)^{n+1} a_{-n}^{(a)\mu} \tag{2.16}$$

Our purpose here is to discuss the definition and the properties of the three strings vertex by exploiting as far as possible the definition given in [14] for the Neumann coefficients. Remembering the description of the star product given in the previous section, the latter is obtained in the following way. Let us consider three unit semidisks in the upper half z_a $(a = 1,2,3)$ plane. Each one represents the string freely propagating in semicircles from the origin (world-sheet time $\tau = -\infty$) to the unit circle $|z_a| = 1$ $(\tau = 0)$, where the interaction is supposed to take place. We map each unit semidisk to a $120°$ wedge of the complex plane via the following conformal maps:

$$f_a(z_a) = \alpha^{2-a} f(z_a)\,,\ a = 1,2,3 \tag{2.17}$$

where

$$f(z) = \Big(\frac{1+iz}{1-iz}\Big)^{\frac{2}{3}} \tag{2.18}$$

Here $\alpha = e^{\frac{2\pi i}{3}}$ is one of the three third roots of unity. In this way the three semidisks are mapped to nonoverlapping (except at the interaction points $z_a = i$) regions in such a way as to fill up a unit disk centered at the origin. The curvature is zero everywhere except at the center of the disk, which represents the common midpoint of the three strings in interaction.

The interaction vertex is defined by a correlation function on the disk in the following way

$$\langle \psi, \phi * \chi\rangle = \langle f_1 \circ \psi(0)\, f_2 \circ \phi(0)\, f_3 \circ \chi(0)\rangle = \langle V_{123}|\psi\rangle_1|\phi\rangle_2|\chi\rangle_3 \tag{2.19}$$

Now we consider the string propagator at two generic points of this disk. The Neumann coefficients N^{ab}_{NM} are nothing but the Fourier modes of the propagator with respect to the original coordinates z_a. We shall see that such Neumann coefficients are related in a simple way to the standard three strings vertex coefficients.

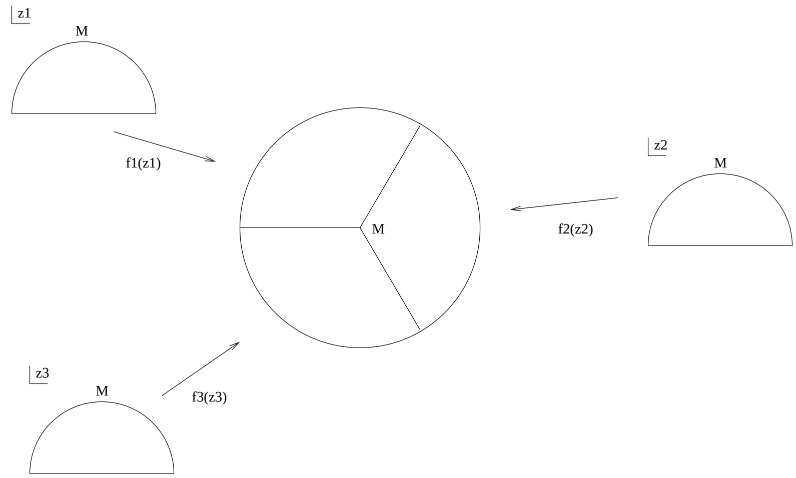

Figure 2.1: *The conformal maps from the three unit semidisks to the three-wedges unit disk*

Due to the qualitative difference between the $\alpha_{n>0}$ oscillators and the zero modes p, the Neumann coefficients involving the latter will be treated separately.

2.1.1 Non–zero modes

The Neumann coefficients N^{ab}_{mn} are given by [14]

$$N^{ab}_{mn} = \langle V_{123}|\alpha^{(a)}_{-n}\alpha^{(b)}_{-m}|0\rangle_{123} = -\frac{1}{nm}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^m}f'_a(z)\frac{1}{(f_a(z)-f_b(w))^2}f'_b(w) \tag{2.20}$$

where the contour integrals are understood around the origin. It is easy to check that

$$\begin{aligned} N^{ab}_{mn} &= N^{ba}_{nm} \\ N^{ab}_{mn} &= (-1)^{n+m}N^{ba}_{mn} \\ N^{ab}_{mn} &= N^{a+1,b+1}_{mn} \end{aligned} \tag{2.21}$$

In the last equation the upper indices are defined mod 3.

Let us consider the decomposition

$$N^{ab}_{mn} = \frac{1}{3\sqrt{nm}}\Big(C_{nm} + \bar{\alpha}^{a-b}U_{nm} + \alpha^{a-b}\bar{U}_{nm}\Big) \tag{2.22}$$

After some algebra one gets

$$\begin{aligned} C_{nm} &= \frac{-1}{\sqrt{nm}}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^m}\Big(\frac{1}{(1+zw)^2}+\frac{1}{(z-w)^2}\Big) \\ U_{nm} &= \frac{-1}{3\sqrt{nm}}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^m}\Big(\frac{f^2(w)}{f^2(z)}+2\frac{f(z)}{f(w)}\Big)\Big(\frac{1}{(1+zw)^2}+\frac{1}{(z-w)^2}\Big) \\ \bar{U}_{nm} &= \frac{-1}{3\sqrt{nm}}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^m}\Big(\frac{f^2(z)}{f^2(w)}+2\frac{f(w)}{f(z)}\Big)\Big(\frac{1}{(1+zw)^2}+\frac{1}{(z-w)^2}\Big) \end{aligned} \tag{2.23}$$

The integrals can be directly computed in terms of the Taylor coefficients of f. The result is

$$\begin{aligned} C_{nm} &= (-1)^n \delta_{nm} && (2.24)\\ U_{nm} &= \frac{1}{3\sqrt{nm}} \sum_{l=1}^{m} l \Big[(-1)^n B_{n-l} B_{m-l} + 2 b_{n-l} b_{m-l} (-1)^m \\ &\quad - (-1)^{n+l} B_{n+l} B_{m-l} - 2 b_{n+l} b_{m-l} (-1)^{m+l} \Big] && (2.25)\\ \bar{U}_{nm} &= (-1)^{n+m} U_{nm} && (2.26) \end{aligned}$$

where we have set

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} b_k z^k \\ f^2(z) &= \sum_{k=0}^{\infty} B_k z^k, \qquad \text{i.e.} \qquad B_k = \sum_{p=0}^{k} b_p b_{k-p} && (2.27) \end{aligned}$$

Eqs.(2.24, 2.25, 2.26) are obtained by expanding the relevant integrands in powers of z, w and correspond to the pole contributions around the origin. We notice that the above integrands have poles also outside the origin, but these poles either are not in the vicinity of the origin of the z and w plane, or, like the poles at $z = w$, simply give vanishing contributions. By changing $z \to -z$ and $w \to -w$, it is easy to show that

$$(-1)^n U_{nm} (-1)^m = \bar{U}_{nm}, \qquad \text{or} \qquad CU = \bar{U}C, \qquad C_{nm} = (-1)^n \delta_{nm} \tag{2.28}$$

In the second part of this equation we have introduced a matrix notation which we will use throughout. One can use this representation for (2.25, 2.26) to make computer calculations. For instance it is easy to show that the equations

$$\sum_{k=1}^{\infty} U_{nk} U_{km} = \delta_{nm}, \qquad \sum_{k=1}^{\infty} \bar{U}_{nk} \bar{U}_{km} = \delta_{nm} \tag{2.29}$$

are satisfied to any desired order of approximation, see the Appendix A for an explicit analytic proof. Each identity follows from the other by using (2.28). It is also easy to make the identification

$$V^{ab}_{nm} = (-1)^{n+m} \sqrt{nm}\, N^{ab}_{nm} \tag{2.30}$$

of the Neumann coefficients with the standard three strings vertex coefficients[1]. Using (2.29), together with the decomposition (2.22), it is easy to establish the commutativity relation (written in matrix notation)

$$[CV^{ab}, CV^{a'b'}] = 0 \tag{2.31}$$

[1]The factor of $(-1)^{n+m}$ in (2.30) arises from the fact that the original definition of the Neumann coefficients (2.20) in [14] refers to the bra three strings vertex $\langle V_3|$, rather than to the ket vertex like in (2.12); therefore the two definitions differ by a bpz operation.

for any a, b, a', b'. This relation is fundamental for the next developments.
It is common to define

$$\begin{aligned} X &= CV^{11} \\ X_+ &= CV^{12} \\ X_- &= CV^{21} \end{aligned} \tag{2.32}$$

Using (2.29), together with the decomposition (2.22), it is easy to establish the following linear and non linear relations (written in matrix notation).

$$\begin{gathered} X + X_+ + X_- = 1 \\ X^2 + X_+^2 + X_-^2 = 1 \\ X_+^3 + X_-^3 = 2X^3 - 3X^2 + 1 \\ X_+ X_- = X^2 - X \\ [X, X_\pm] = 0 \\ [X_+, X_-] = 0 \end{gathered} \tag{2.33}$$

These very important properties encode the associativity of the matter star product.

2.1.2 Zero modes

The Neumann coefficients involving one zero mode are given by

$$N_{0m}^{ab} = -\frac{1}{m} \oint \frac{dw}{2\pi i} \frac{1}{w^m} f_b'(w) \frac{1}{f_a(0) - f_b(w)} \tag{2.34}$$

In this case too we make the decomposition

$$N_{0m}^{ab} = \frac{1}{3} \Big(E_m + \bar{\alpha}^{a-b} U_m + \alpha^{a-b} \bar{U}_m \Big) \tag{2.35}$$

where $E, U, \bar{U}$ can be given, after some algebra, the explicit expression

$$\begin{aligned} E_n &= -\frac{4i}{n} \oint \frac{dw}{2\pi i} \frac{1}{w^n} \frac{1}{1+w^2} \frac{f^3(w)}{1 - f^3(w)} = \frac{2i^n}{n} \\ U_n &= -\frac{4i}{n} \oint \frac{dw}{2\pi i} \frac{1}{w^n} \frac{1}{1+w^2} \frac{f^2(w)}{1 - f^3(w)} = \frac{\alpha_n}{n} \\ \bar{U}_n &= (-1)^n U_n = (-1)^n \frac{\alpha_n}{n} \end{aligned} \tag{2.36}$$

The numbers α_n are Taylor coefficients

$$\sqrt{f(z)} = \sum_0^\infty \alpha_n z^n$$

They are related to the A_n coefficients of Appendix B of [54] (see also [52]) as follows: $\alpha_n = A_n$ for n even and $\alpha_n = iA_n$ for n odd. N_{0n}^{ab} are not related in a simple way as (2.30) to the corresponding three strings vertex coefficients. The reason is that the latter satisfy the conditions

$$\sum_{a=1}^{3} V_{0n}^{ab} = 0 \tag{2.37}$$

These constraints fix the invariance $V_{0n}^{ab} \to V_{0n}^{ab} + B_n^b$, where B_n^b are arbitrary numbers, an invariance which arises in the vertex (2.12) due to momentum conservation. For the Neumann coefficients N_{0n}^{ab} we have instead

$$\sum_{a=1}^{3} V_{0n}^{ab} = E_n \tag{2.38}$$

It is thus natural to define

$$\hat{N}_{0n}^{ab} = N_{0n}^{ab} - \frac{1}{3} E_n \tag{2.39}$$

Now one can easily verify that[2]

$$V_{0n}^{ab} = -\sqrt{2n}\, \hat{N}_{0n}^{ab} \tag{2.40}$$

It is somewhat surprising that in this relation we do not meet the factor $(-1)^n$, which we would expect on the basis of the *bpz* conjugation (see footnote after eq.(2.30)). However eq.(2.40) is also naturally requested by the integrable structure found in [19]. The absence of the $(-1)^n$ factor corresponds to the exchange $V_{0n}^{12} \leftrightarrow V_{0n}^{21}$. This exchange does not seem to affect in any significant way the results obtained so far in this field.

To complete the discussion about the matter sector one should recall that beside eq.(2.29), there are other basic equations from which all the results about the Neumann coefficients can be derived. They concern the quantities

$$W_n = -\sqrt{2n}\, U_n = -\sqrt{\frac{2}{n}}\, \alpha_n, \qquad W_n^* = -\sqrt{2n}\, \bar{U}_n = -\sqrt{\frac{2}{n}}\, (-1)^n \alpha_n \tag{2.41}$$

The relevant identities, [52, 54], are

$$\sum_{n=1}^{\infty} W_n\, U_{np} = W_p, \qquad \sum_{n\geq 1} W_n W_n^* = 2V_{00}^{aa} \tag{2.42}$$

These identities can easily be shown numerically to be correct at any desired approximation.

Finally let us concentrate on the Neumann coefficients N_{00}^{ab}. Although a formula for them can be found in [14], these numbers are completely arbitrary due to momentum conservation. The choice

$$V_{00}^{ab} = \delta_{ab} \ln \frac{27}{16} \tag{2.43}$$

[2] The $\sqrt{2}$ factor is there because in [54] the $\alpha' = 1$ convention is used

is the same as in [52], but it is also motivated by one of the most surprising and mysterious aspects of SFT, namely its underlying integrable structure: the matter Neumann coefficients obey the Hirota equations of the dispersionless Toda lattice hierarchy. This was explained in [19] following a suggestion of [20]. On the basis of these equations the matter Neumann coefficients with nonzero labels can be expressed in terms of the remaining ones. The choice of (2.43) in this context is natural.

2.1.3 Oscillator representation of the zero modes

In many computations we will deal with object that are localized along some directions, such as lower dimensional D–branes. Therefore translational invariance will be broken and the momentum will not be anymore a good variable as far as zero modes are concerned. We will therefore use another basis for the zero mode x, p.

First we split the Lorentz indices μ, ν into parallel ones, $\bar{\mu}, \bar{\nu}$, running from 0 to $25-k$, and transverse ones, α, β which run from $26-k$ to 25. Next we introduce the new zero modes by defining

$$a_0^{(r)\alpha} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} - i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}, \qquad a_0^{(r)\alpha\dagger} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} + i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}, \tag{2.44}$$

where $\hat{p}^{(r)\alpha}, \hat{x}^{(r)\alpha}$ are the momentum and position operator of the r–th string. The parameter b is as in ref.[54]. The Dirac brackets for all the oscillators including the zero modes are, in the transverse directions,

$$[a_M^{(r)\alpha}, a_N^{(s)\beta\dagger}] = \eta^{\alpha\beta}\delta^{rs}\delta_{MN}, \qquad\qquad N, M \geq 0 \tag{2.45}$$

where the index N denotes the couple $(0, n)$. Now we introduce $|\Omega_b\rangle$, the oscillator vacuum ($a_N^\alpha|\Omega_b\rangle = 0$, for $N \geq 0$). The relation between the momentum basis and the oscillator basis is defined by

$$|\{p^\alpha\}\rangle_{123} = \left(\frac{b}{2\pi}\right)^{\frac{k}{4}} e^{\sum_{r=1}^3 \left(-\frac{b}{4}p_\alpha^{(r)}\eta^{\alpha\beta}p_\beta^{(r)} + \sqrt{b}a_0^{(r)\alpha\dagger}p_\alpha^{(r)} - \frac{1}{2}a_0^{(r)\alpha\dagger}\eta_{\alpha\beta}a_0^{(r)\beta\dagger}\right)}|\Omega_b\rangle$$

Inserting this into (2.12) and integrating with respect to the transverse momenta one finally gets the following three strings vertex [52, 54]

$$|V_3\rangle' = |V_{3,\perp}\rangle' \otimes |V_{3,\|}\rangle \tag{2.46}$$

$|V_{3,\|}\rangle$ is the one used before this subsection , while

$$|V_{3,\perp}\rangle' = K_2\, e^{-E'}|\Omega_b\rangle \tag{2.47}$$

where K_2 is a suitable constant and

$$E' = \frac{1}{2}\sum_{r,s=1}^3 \sum_{M,N\geq 0} a_M^{(r)\alpha\dagger} {V'}^{rs}_{MN} a_N^{(s)\beta\dagger}\eta_{\alpha\beta} \tag{2.48}$$

The vertex coefficients ${V'}^{rs}_{MN}$ to be used in the transverse directions have parallel properties to the vertex V^{rs}_{mn}. When multiplied by the twist matrix they give rise to matrices X', X'_+, X'_- which happen to obey the same equations collected in Appendix A for the matrices X, X_+, X_-

2.2 Ghost three strings vertex and bc Neumann coefficients

The three strings vertex for the ghost part is more complicated than the matter part due to the zero modes of the c field. As we will see, the latter generate an ambiguity in the definition of the Neumann coefficients. Such an ambiguity can however be exploited to formulate and solve in a compact form the problem of finding solutions to eq.(4.5)

2.2.1 Neumann coefficients: definitions and properties

To start with we define, in the ghost sector, the vacuum states $|\hat{0}\rangle$ and $|\dot{0}\rangle$ as follows

$$|\hat{0}\rangle = c_0 c_1 |0\rangle, \qquad |\dot{0}\rangle = c_1 |0\rangle \tag{2.49}$$

where $|0\rangle$ is the usual $SL(2,\mathbb{R})$ invariant vacuum. Using bpz conjugation

$$c_n \to (-1)^{n+1} c_{-n}, \qquad b_n \to (-1)^{n-2} b_{-n}, \qquad |0\rangle \to \langle 0| \tag{2.50}$$

one can define conjugate states. It is important that, when applied to products of oscillators, the bpz conjugation does not change the order of the factors, but transforms rigidly the vertex and all the squeezed states we will consider in the sequel (see for instance eq.(2.52) below).

The three strings interaction vertex is defined, as usual, as a squeezed operator acting on three copies of the bc Hilbert space

$$\langle \tilde{V}_3| = {}_1\langle\hat{0}|\, {}_2\langle\hat{0}|\, {}_3\langle\hat{0}| e^{\tilde{E}}, \qquad \tilde{E} = \sum_{a,b=1}^{3} \sum_{n,m}^{\infty} c_n^{(a)} \tilde{N}^{ab}_{nm} b_m^{(b)} \tag{2.51}$$

Under bpz conjugation

$$|\tilde{V}_3\rangle = e^{\tilde{E}'} |\hat{0}\rangle_1 |\hat{0}\rangle_2 |\hat{0}\rangle_3, \qquad \tilde{E}' = -\sum_{a,b=1}^{3} \sum_{n,m}^{\infty} (-1)^{n+m} c_n^{(a)\,\dagger} \tilde{N}^{ab}_{nm} b_m^{(b)\,\dagger} \tag{2.52}$$

In eqs.(2.51, 2.52) we have not specified the lower bound of the m, n summation. This point will be clarified below.

The Neumann coefficients $\tilde{N}^{ab}_{nm}$ are given by the contraction of the bc oscillators on the unit disk (constructed out of three unit semidisks, as explained in section 3). They represent Fourier components of the $SL(2,\mathbb{R})$ invariant bc propagator (i.e. the propagator in which the zero mode have been inserted at fixed points ζ_i, $i = 1, 2, 3$):

$$\langle b(z) c(w) \rangle = \frac{1}{z-w} \prod_{i=1}^{3} \frac{w-\zeta_i}{z-\zeta_i} \tag{2.53}$$

Taking into account the conformal properties of the b, c fields we get

$$
\begin{aligned}
\tilde{N}^{ab}_{nm} &= \langle \tilde{V}_{123} | b^{(a)}_{-n} c^{(b)}_{-m} | \dot{0} \rangle_{123} \\
&= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n-1}} \frac{1}{w^{m+2}} (f'_a(z))^2 \frac{-1}{f_a(z) - f_b(w)} \prod_{i=1}^{3} \frac{f_b(w) - \zeta_i}{f_a(z) - \zeta_i} (f'_b(w))^{-1}
\end{aligned} \tag{2.54}
$$

It is straightforward to check that

$$
\tilde{N}^{ab}_{nm} = \tilde{N}^{a+1,b+1}_{nm} \tag{2.55}
$$

and (by letting $z \to -z$, $w \to -w$)

$$
\tilde{N}^{ab}_{nm} = (-1)^{n+m} \tilde{N}^{ba}_{nm} \tag{2.56}
$$

Now we choose $\zeta_i = f_i(0) = \alpha^{2-i}$ so that the product factor in (2.54) nicely simplifies as follows

$$
\prod_{i=1}^{3} \frac{f_b(w) - f_i(0)}{f_a(z) - f_i(0)} = \frac{f^3(w) - 1}{f^3(z) - 1}, \qquad \forall\, a, b = 1, 2, 3 \tag{2.57}
$$

Now, as in the matter case, we consider the decomposition

$$
\tilde{N}^{ab}_{nm} = \frac{1}{3}(\tilde{E}_{nm} + \bar{\alpha}^{a-b} \tilde{U}_{nm} + \alpha^{a-b} \bar{\tilde{U}}_{nm}) \tag{2.58}
$$

After some elementary algebra, using $f'(z) = \frac{4i}{3} \frac{1}{1+z^2} f(z)$, one finds

$$
\begin{aligned}
\tilde{E}_{nm} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^{m+1}} \left(\frac{1}{1+zw} - \frac{w}{w-z} \right) \\
\tilde{U}_{nm} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^{m+1}} \frac{f(z)}{f(w)} \left(\frac{1}{1+zw} - \frac{w}{w-z} \right) \\
\bar{\tilde{U}}_{nm} &= \oint \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{z^{n+1}} \frac{1}{w^{m+1}} \frac{f(w)}{f(z)} \left(\frac{1}{1+zw} - \frac{w}{w-z} \right)
\end{aligned} \tag{2.59}
$$

Using the property $f(-z) = (f(z))^{-1}$, one can easily prove that

$$
\bar{\tilde{U}}_{nm} = (-1)^{n+m} \tilde{U}_{nm} \tag{2.60}
$$

2.2.2 Computation of the coefficients

In this section we explicitly compute the above integrals. We shall see that the presence of the three c zero modes induces an ambiguity in the $(0,0)$, $(-1,1)$, $(1,-1)$ components of the Neumann coefficients. This in turn arises from the ambiguity in the radial ordering of the integration variables z, w. While the result does not depend on what variable we integrate first, it does depend in general on whether $|z| > |w|$ or $|z| < |w|$.

If we choose $|z| > |w|$ we get

$$
\tilde{E}^{(1)}_{nm} = \theta(n)\theta(m)(-1)^n \delta_{nm} + \delta_{n,0}\delta_{m,0} + \delta_{n,-1}\delta_{m,1} \tag{2.61}
$$

while, if we choose $|z| < |w|$, we obtain

$$\tilde{E}^{(2)}_{nm} = \theta(n)\theta(m)(-1)^n\delta_{nm} - \delta_{n,1}\delta_{m,-1} \tag{2.62}$$

where $\theta(n) = 1$ for $n > 0$, $\theta(n) = 0$ for $n \leq 0$. We see that the result is ambiguous for the components $(0,0)$, $(-1,1)$, $(1,-1)$.

To compute $\tilde{U}_{nm}$ we expand $f(z)$ for small z, as in section 3,

$$f(z) \quad = \quad \sum_{k=0}^{\infty} b_k z^k$$

Since $f^{-1}(z) = f(-z)$ we get the relation

$$\sum_{k=0}^{n}(-1)^k b_k b_{n-k} = \delta_{n,0} \tag{2.63}$$

which is identically satisfied for n odd, while for n even it can be also rewritten as

$$b_n^2 = -2\sum_{k=1}^{n}(-1)^k b_{n-k}b_{n+k} \tag{2.64}$$

Taking $|z| > |w|$ and integrating z first, one gets

$$\tilde{U}^{(1a)}_{nm} = \delta_{n+m} + (-1)^m \sum_{l=1}^{n}(b_{n-l}b_{m-l} - (-1)^l b_{n-l}b_{m+l}) \tag{2.65}$$

If, instead, we integrate w first,

$$\tilde{U}^{(1b)}_{nm} = (-1)^m b_n b_m + (-1)^m \sum_{l=1}^{m}(b_{n-l}b_{m-l} + (-1)^l b_{n+l}b_{m-l}) \tag{2.66}$$

One can check that, due to (2.64),

$$\tilde{U}^{(1a)}_{nm} \quad = \quad \tilde{U}^{(1b)}_{nm} \equiv \tilde{U}^{(1)}_{nm} \tag{2.67}$$

Now we take $|z| < |w|$ and get similarly

$$\tilde{U}^{(2a)}_{nm} = (-1)^m \sum_{l=1}^{n}(b_{n-l}b_{m-l} - (-1)^l b_{n-l}b_{m+l})$$

$$\tilde{U}^{(2b)}_{nm} = -\delta_{n+m} + (-1)^m b_n b_m + (-1)^m \sum_{l=1}^{m}(b_{n-l}b_{m-l} + (-1)^l b_{n+l}b_{m-l})$$

Again, due to (2.64)

$$\tilde{U}^{(2a)}_{nm} \quad = \quad \tilde{U}^{(2b)}_{nm} = \tilde{U}^{(2)}_{nm} \tag{2.68}$$

Comparing $\tilde{U}^{(1)}$ with $\tilde{U}^{(2)}$, we see once more that the ambiguity only concerns the $(0,0)$, $(-1,1)$, $(1,-1)$ components. Using (2.58) we define

$$\tilde{N}^{ab,(1,2)}_{nm} \quad = \quad \frac{1}{3}(\tilde{E}^{(1,2)}_{nm} + \bar{\alpha}^{(a-b)}\tilde{U}^{(1,2)}_{nm} + \alpha^{a-b}(-1)^{n+m}\tilde{U}^{(1,2)}_{nm})$$

The above ambiguity propagates also to these coefficients, but only when $a = b$. For later reference it is useful to notice that

$$\begin{aligned}\tilde{N}^{ab,\,(1,2)}_{-1,m} &= 0, \quad \text{except} \quad \text{perhaps} \quad \text{for} \quad a = b, \quad m = 1 \\ \tilde{N}^{ab,\,(1,2)}_{0,m} &= 0, \quad \text{except} \quad \text{perhaps} \quad \text{for} \quad a = b \quad m = 0\end{aligned} \tag{2.69}$$

and, for $|n| \leq 1$,

$$\tilde{N}^{ab,\,(1,2)}_{n,1} = 0, \quad \text{except} \quad \text{perhaps} \quad \text{for} \quad a = b \quad n = -1 \tag{2.70}$$

We notice that, if in eq.(2.51,2.52) the summation over m, n starts from -1, the above ambiguity is consistent with the general identification proposed in [14]

$$\tilde{N}^{ab}_{nm} = \langle \tilde{V}_3 | b^{(a)}_{-n} c^{(b)}_{-m} |\dot{0}\rangle_1 |\dot{0}\rangle_2 |\dot{0}\rangle_3 \tag{2.71}$$

It is easy to see that the expression in the RHS is not *bpz* covariant when (m, n) take values $(0, 0)$, $(-1, 1)$, $(1, -1)$ and the lower bound of the m, n summation in the vertex (see above) is -1. Such *bpz* noncovariance corresponds exactly to the ambiguity we have come across in the explicit evaluation of the Neumann coefficients. We can refer to it as the *bpz* or *radial ordering anomaly.*

2.2.3 Two alternatives

It is clear that we are free to fix the ambiguity the way we wish, provided the convention we choose is consistent with *bpz* conjugation. We consider here two possible choices. The first consists in setting to zero all the components of the Neumann coefficients which are ambiguous, i.e. the $(0, 0)$, $(-1, 1)$, $(1, -1)$ ones. This leads to a definition of the vertex (2.51) in which the summation over n starts from 1 while the summation over m starts from 0. In this way any ambiguity is eliminated and the Neumann coefficients are *bpz* covariant. This is the preferred choice in the literature, [22, 24, 23, 25, 26]. In particular, it has led in [22] to a successful comparison of the operator formulation with a twisted conformal field theory one.

We would like, now, to make some comments about this first choice, with the purpose of stressing the difference with the alternative one we will discuss next. In particular we would like to anticipate some aspects of the BRST cohomology in Vacuum String Field Theory (VSFT). In VSFT the BRST operator is conjectured [22, 23] to take the form

$$\mathcal{Q} = c_0 + \sum_{n=1}^{\infty} f_n (c_n + (-1)^n c_{-n}) \tag{2.72}$$

It is easy to show that the vertex is BRST invariant (Q is a derivation of the $*$–product), i.e.

$$\sum_{a=1}^{3} \mathcal{Q}^{(a)} |\tilde{V}_3\rangle = 0 \tag{2.73}$$

Due to

$$\{\mathcal{Q}, b_0\} = 1 \tag{2.74}$$

it follows that the cohomology of $\mathcal{Q}$ is trivial. As was noted in [25], this implies that the subset of the string field algebra is the direct sum of $\mathcal{Q}$–closed states and b_0–closed states (i.e. states in the Siegel gauge).

$$|\Psi\rangle = \mathcal{Q}|\lambda\rangle + b_0|\chi\rangle \tag{2.75}$$

As a consequence of the BRST invariance of the vertex it follows that the star product of a BRST-exact state with any other is identically zero.

For this reason previous calculations were done with the use of the *reduced* vertex [23, 22] which consists of Neumann coefficients starting from the (1,1) component. The reduced product is explicitly defined by

$$|\psi *_{b_0} \phi\rangle = b_0|\psi * \phi\rangle \tag{2.76}$$

Note that this product, at ghost number 1, does not increase the ghost number.

The unreduced star product can be recovered by the midpoint insertion of $\mathcal{Q} = \frac{1}{2i}(c(i) - c(-i))$ as

$$|\psi * \phi\rangle = \mathcal{Q}|\psi *_{b_0} \phi\rangle \tag{2.77}$$

In the alternative treatment given below, using an enlarged Fock space, we compute the star product without any gauge choice and any explicit midpoint insertion.

Motivated by the advantages it offers in the search of solutions to (4.5), we propose therefore a second option. It consists in fixing the ambiguity by setting

$$\tilde{N}^{aa}_{-1,1} = \tilde{N}^{aa}_{1,-1} = 0, \qquad \tilde{N}^{aa}_{0,0} = 1. \tag{2.78}$$

If we do so we get a fundamental identity, valid for $\tilde{U}_{nm} \equiv \tilde{U}^{(1)}_{nm}$ (for $n, m \geq 0$),

$$\sum_{k=0} \tilde{U}_{nk}\tilde{U}_{km} = \delta_{nm} \tag{2.79}$$

Defining

$$\tilde{X}^{ab} = C\tilde{V}^{ab}, \tag{2.80}$$

eq.(2.79) entails

$$[\tilde{X}^{ab}, \tilde{X}^{a'b'}] = 0 \tag{2.81}$$

One can prove eq.(2.79) numerically. By using a cutoff in the summation one can approximate the result to any desired order (although the convergence with increasing cutoff is less rapid than in the corresponding matter case, see section 3.1). A direct analytic proof of eq.(2.79) is given in Appendix.

The next subsection is devoted to working out some remarkable consequences of eq.(2.79).

2.2.4 Matrix structure

Once the convention (2.78) is chosen, we recognize that all the matrices $(\tilde{E}, \tilde{U}, \bar{\tilde{U}})$ have the $(0,0)$ component equal to 1, all the other entries of the upper row equal to 0, and a generally non vanishing zeroth column. More precisely

$$\begin{aligned} &\tilde{U}_{00} = \tilde{E}_{00} = 1 \\ &\tilde{U}_{n0} = b_n \quad \tilde{E}_{n0} = 0, \qquad \tilde{U}_{0n} = \tilde{E}_{0n} = \delta_{n,0} \\ &\tilde{U}_{nm} \neq 0, \quad n, m > 0 \end{aligned} \tag{2.82}$$

This particular structure makes this kind of matrices simple to handle under a generic analytic map f. In order to see this, let us inaugurate a new notation, which we will use in this and the next section. We recall that the labels M, N indicate the couple $(0, m), (0, n)$. Given a matrix M, let us distinguish between the 'large' matrix M_{MN} denoted by the calligraphic symbol $\mathcal{M}$ and the 'small' matrix M_{mn} denoted by the plain symbol M. Accordingly, we will denote by $\mathcal{Y}$ a matrix of the form (2.82), $\vec{y} = (y_1, y_2, ...)$ will denote the nonvanishing column vector and Y the 'small' matrix

$$\mathcal{Y}_{NM} = \delta_{N0}\delta_{M0} + y_n\delta_{M0} + Y_{mn}, \tag{2.83}$$

or, symbolically, $\mathcal{Y} = (1, \vec{y}, Y)$.

Then, using a formal Taylor expansion for f, one can show that

$$f[\mathcal{Y}]_{NM} = f[1]\delta_{N0}\delta_{M0} + \Big(\frac{f[1] - f[Y]}{1 - Y}\vec{y}\Big)_n \delta_{M0} + f[Y]_{mn} \tag{2.84}$$

Now let us define

$$\begin{aligned} \mathcal{Y} &\equiv \tilde{X}^{11} \\ \mathcal{Y}_+ &\equiv \tilde{X}^{12} \qquad (2.85) \\ \mathcal{Y}_- &\equiv \tilde{X}^{21} \qquad (2.86) \end{aligned}$$

These three matrices have the above form. Using (2.79) one can prove the following properties (which are well–known for the 'small' matrices)

$$\begin{aligned} &\mathcal{Y} + \mathcal{Y}_+ + \mathcal{Y}_- = 1 \\ &\mathcal{Y}^2 + \mathcal{Y}_+^2 + \mathcal{Y}_-^2 = 1 \\ &\mathcal{Y}_+^3 + \mathcal{Y}_-^3 = 2\mathcal{Y}^3 - 3\mathcal{Y}^2 + 1 \\ &\mathcal{Y}_+\mathcal{Y}_- = \mathcal{Y}^2 - \mathcal{Y} \\ &[\mathcal{Y}, \mathcal{Y}_\pm] = 0 \\ &[\mathcal{Y}_+, \mathcal{Y}_-] = 0 \end{aligned} \tag{2.87}$$

Using (2.83, 2.84) we immediately obtain (we point out that, in particular for $\mathcal{Y}$, $y_{2n} = \frac{2}{3}\, b_{2n}$, $y_{2n+1} = 0$ and $Y_{nm} = \tilde{X}_{nm}$ for $n, m > 0$)

$$
\begin{aligned}
Y + Y_+ + Y_- &= 1 \\
\vec{y} + \vec{y}_+ + \vec{y}_- &= 0 \\
Y^2 + Y_+^2 + Y_-^2 &= 1 \\
(1+Y)\vec{y} + Y_+\vec{y}_+ + Y_-\vec{y}_- &= 0 \\
Y_+^3 + Y_-^3 &= 2Y^3 - 3Y^2 + 1 \\
Y_+^2\vec{y}_+ + Y_-^2\vec{y}_- &= (2Y^2 - Y - 1)\vec{y} \\
Y_+Y_- &= Y^2 - Y \\
[Y, Y_\pm] &= 0 \\
[Y_+, Y_-] &= 0 \\
Y_+\vec{y}_- = Y\vec{y} &= Y_-\vec{y}_+ \\
-Y_\pm\vec{y} &= (1-Y)\vec{y}_\pm
\end{aligned}
\tag{2.88}
$$

These properties were shown in various papers, see [23, 26]. Here they are simply consequences of (2.87), and therefore of (2.79). In particular we note that the properties of the 'large' matrices are isomorphic to those of the 'small' ones. This fact allows us to work directly with the 'large' matrices, handling at the same time both zero and not zero modes.

2.2.5 Enlarged Fock space

We have seen in the last subsection the great advantages of introducing the convention (2.78). In this subsection we make a proposal as to how to incorporate this convention in an enlargement of the bc system's Fock space. In fact, in order for eq.(2.71) to be consistent, a modification in the RHS of this equation is in order. This can be done by, so to speak, 'blowing up' the zero mode sector. We therefore enlarge the original Fock space, while warning that our procedure may be far from unique. For each string, we split the modes c_0 and b_0:

$$\eta_0 \leftarrow c_0 \rightarrow \eta_0^\dagger, \qquad \xi_0^\dagger \leftarrow b_0 \rightarrow \xi_0 \tag{2.89}$$

In other words we introduce two additional couple of conjugate anticommuting creation and annihilation operators $\eta_0, \eta_0^\dagger$ and $\xi_0, \xi_0^\dagger$

$$\{\xi_0, \eta_0\} = 1, \qquad \{\xi_0^\dagger, \eta_0^\dagger\} = 1 \tag{2.90}$$

with the following rules on the vacuum

$$\xi_0|0\rangle = 0, \qquad \langle 0|\xi_0^\dagger = 0 \tag{2.91}$$

$$\eta_0^\dagger|0\rangle = 0, \qquad \langle 0|\eta_0 = 0 \tag{2.92}$$

while $\xi_0^\dagger, \eta_0$ acting on $|0\rangle$ create new states. The *bpz* conjugation properties are defined by

$$bpz(\eta_0) = -\eta_0^\dagger, \qquad bpz(\xi_0) = \xi_0^\dagger \tag{2.93}$$

The reason for this difference is that η_0 (ξ_0) is meant to be of the same type as c_0 (b_0). The anticommutation relation of c_0 and b_0 remain the same

$$\{c_0, b_0\} = 1 \tag{2.94}$$

All the other anticommutators among these operators and with the other *bc* oscillators are required to vanish. In the enlarged Fock space all the objects we have defined so far may get slightly changed. In particular the three strings vertex (2.51,2.52) is now defined by

$$\tilde{E}'_{(en)} = \sum_{n\geq 1, m\geq 0}^{\infty} c_n^{(a)\,\dagger} \tilde{V}_{nm}^{(ab)} b_m^{(b)\dagger} - \eta_0^{(a)} b_0^{(a)} \tag{2.95}$$

With this redefinition of the vertex any ambiguity is eliminated, as one can easily check. In a similar way we may have to modify all the objects that enter into the game.

The purpose of the Fock space enlargement is to make us able to evaluate vev's of the type

$$\langle\dot{0}|\exp\Big(cFb + c\mu + \lambda b\Big)\exp\Big(c^\dagger G b^\dagger + \theta b^\dagger + c^\dagger\zeta\Big)|\hat{0}\rangle \tag{2.96}$$

which are needed in star products. Here we use an obvious compact notation: F, G denotes matrices F_{NM}, G_{NM}, and $\lambda, \mu, \theta, \zeta$ are anticommuting vectors $\lambda_N, \mu_N, \theta_N, \zeta_N$. In $cFb + c\mu + \lambda b$ it is understood that the mode b_0 is replaced by ξ_0 and in $c^\dagger G b^\dagger + \theta b^\dagger + c^\dagger \zeta$ the mode c_0 is replaced by η_0. In this way the formula is unambiguous and we obtain

$$\langle\dot{0}|\exp\left(cFb + c\mu + \lambda b\right)\exp\left(c^\dagger G b^\dagger + \theta b^\dagger + c^\dagger\zeta\right)|\hat{0}\rangle$$

$$= \det(1+FG)\exp\left(-\theta\tfrac{1}{1+FG}F\zeta - \lambda\tfrac{1}{1+GF}G\mu - \theta\tfrac{1}{1+FG}\mu + \lambda\tfrac{1}{1+GF}\zeta\right) \tag{2.97}$$

Eventually, after performing the star products, we will return to the original Fock space.

2.2.6 The twisted star

In [22] another type of star-product is considered. It represents the gluing condition in a twisted conformal field theory of the ghost system. The twist is done by subtracting to the stress tensor one unit of derivative of the ghost current

$$T'(z) = T(z) - \partial j_{gh}(z) \tag{2.98}$$

This redefinition changes the conformal weight of the *bc* fields from (2,-1) to (1,0). It follows that the background charge is shifted from -3 to -1. As a consequence, in order not to have vanishing correlation functions, we have to fix only one c zero-mode. In particular, the $SL(2,\mathbb{R})$–invariant propagator of the bc system is

$$\langle b(z)c(w)\rangle' = \frac{1}{z-w}\frac{w-\xi}{z-\xi} \tag{2.99}$$

where ξ is one fixed point.

In [22] it was shown that the usual product can be obtained from the twisted one by inserting a $n_{gh}=1$–operator at the midpoint which, on singular states like the sliver (see next sections), can be identified with a c–midpoint insertion. This implies that, on such singular projectors, the twisted product can be identified with the reduced one.

The twisted ghost Neumann coefficients are then defined to be[3]

$$\begin{aligned}\tilde{N}'^{ab}_{nm} &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^{m+1}} f'_a(z)\frac{-1}{f_a(z)-f_b(w)}\frac{f_b(w)}{f_a(z)} \\ &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^{m+1}}\frac{4i}{3}\frac{1}{1+z^2}\frac{\bar{\alpha}^b f(w)}{\bar{\alpha}^a f(z)-\bar{\alpha}^b f(w)}\end{aligned} \tag{2.100}$$

As in (2.54) these coefficients refer to the Bra vertex, the corresponding coefficients for the Ket vertex are

$$\tilde{V}'^{ab}_{nm} = -(-1)^{n+m}\tilde{N}'^{ab}_{nm} \tag{2.101}$$

We will see in the next section how to compute such coefficients using previous results. This will lead to interesting connections with the other star-products.

2.3 Relations among the stars

In this section we will show how the stars products defined above are related to each other. In particular we will show the explicit relations which connect all the Neumann coefficients in the game.

2.3.1 Twisted ghosts vs Matter

The commuting matter Neumann coefficients which appear in (2.33) are given by

$$X^{ab}_{nm} = -\frac{(-1)^m}{\sqrt{nm}}\oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^m} f'_a(z)\frac{1}{(f_a(z)-f_b(w))^2} f'_b(w) \tag{2.102}$$

We can rewrite them as

$$\begin{aligned}X^{ab}_{nm} &= -\frac{(-1)^m}{\sqrt{nm}}\oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^m} f'_a(z)\partial_w\frac{1}{f_a(z)-f_b(w)} \\ &= -(-1)^m\sqrt{\frac{m}{n}}\oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^{m+1}}\frac{f'_a(z)}{f_a(z)-f_b(w)}\end{aligned} \tag{2.103}$$

[3]We put, for simplicity, $\xi = f_a(i) = 0$

where we have integrated by part to respect the variable w. Now, recalling

$$f'_a(z) = \frac{4i}{3}\frac{1}{1+z^2}\alpha^{2-a}f(z)\ , \tag{2.104}$$

we obtain

$$X^{ab}_{nm} = -(-1)^m\sqrt{\frac{m}{n}}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^{m+1}}\frac{4i}{3}\frac{1}{1+z^2}\frac{\bar{\alpha}^a f(z)}{\bar{\alpha}^a f(z)-\bar{\alpha}^b f(w)} \tag{2.105}$$

Let us now consider the corresponding twisted ghost Neumann coefficients

$$\begin{aligned} Y'^{ab}_{nm} &= (C\tilde{V}'^{ab})_{nm} \\ &= (-1)^m\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^{m+1}}\frac{f'_a(z)}{(f_a(z)-f_b(w))}\frac{f_b(w)}{f_a(z)} \\ &= (-1)^m\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^{m+1}}\frac{4i}{3}\frac{1}{1+z^2}\frac{\bar{\alpha}^b f(w)}{(\bar{\alpha}^a f(z)-\bar{\alpha}^b f(w))} \end{aligned} \tag{2.106}$$

This coefficients are not symmetric if we exchange n with m, however we can easily symmetrize them by the use of the matrix $E_{nm} = \sqrt{n}\delta_{nm}$

$$Y'^{ab} \to E^{-1}Y'^{ab}E \tag{2.107}$$

It is now easy to show the following

$$\begin{aligned} (E^{-1}Y'^{ab}E)_{nm} + X^{ab}_{nm} &= (-1)^m\sqrt{\frac{m}{n}}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^{m+1}}\frac{4i}{3}\frac{1}{1+z^2}\frac{(\bar{\alpha}^b f(w)-\bar{\alpha}^a f(z))}{(\bar{\alpha}^a f(z)-\bar{\alpha}^b f(w))} \\ &= -(-1)^m\sqrt{\frac{m}{n}}\oint\frac{dz}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{z^n}\frac{1}{w^{m+1}}\frac{4i}{3}\frac{1}{1+z^2} = 0 \end{aligned} \tag{2.108}$$

the last equality holding since there are no poles for $n, m \geq 1$.
So we obtain

$$E^{-1}Y'^{ab}E = -X^{ab} \tag{2.109}$$

a remarkable relation between twisted ghost and matter vertices, which is the same relation that holds in the four-string vertex between the non-twisted ghost and the matter Neumann coefficients [53]. This relation proves also that the ghost integral is independent of the background charge, for $n, m \geq 1$: the matter integral, indeed, can be seen as the ghost integral without the background charge[4]. As a consequence of the relation with the matter coefficients we can derive all the relevant properties of the twisted ghost Neumann coefficients, by simply taking the matter results (2.33) and changing the sign in odd powers.

$$Y' + Y'_+ + Y'_- = -1$$
$$Y'^2 + Y'^2_+ + Y'^2_- = 1$$

[4]The independence of the background charge is also crucial to prove $\tilde{N}'^{ab} = C\tilde{N}'^{ba}C$

$$
\begin{aligned}
Y_+'^3 + Y_-'^3 &= 2Y'^3 + 3Y'^2 - 1 \\
Y_+' Y_-' &= Y'^2 + Y' \\
[Y', Y_\pm'] &= 0 \\
[Y_+', Y_-'] &= 0
\end{aligned}
\tag{2.110}
$$

2.3.2 Twisted vs Reduced

The relation between the twisted and non-twisted ghost Neumann coefficients can now be obtained using the previous relation

$$
Y' = -EXE^{-1} \tag{2.111}
$$

and the Gross-Jevicki relation [53]

$$
Y = E\frac{-X}{1+2X}E^{-1} \tag{2.112}
$$

between matter and non-twisted ghosts. So, finally, we have

$$
Y = \frac{Y'}{1-2Y'} \tag{2.113}
$$

or

$$
Y' = \frac{Y}{1+2Y} \tag{2.114}
$$

This relation is also strictly related to the equality of solutions between the ghost sliver constructed from the twisted CFT and the non-twisted one [56]. Indeed, it is possible to derive such relation from the equality of ghost algebraic slivers, as we will see in the next section.

2.4 Diagonalization of the Neumann coefficients

We have seen that the star product is encoded in the infinite dimensional Neumann matrices. Although every entry of such matrices has been computed in the previous section, it is in general not easy to deal with them in explicit computations. One should however notice that all these matrices are symmetric and real, so they can be diagonalized with real eigenvalues. The knowledge of eigenvalues and eigenvectors will allow us to evaluate exactly many quantities related to physical observables that otherwise would have been computable only (and numerically) with a finite level truncation of the infinite dimensional matrices. The remaining part of this chapter is devoted to the explicit spectroscopy of all the kinds of star product we have so far analyzed.

2.4.1 Spectroscopy and diagonal representation in the matter sector

The diagonalization of the X matrix was carried out in [28], while the same analysis for X' was accomplished in [79] and [50]. Here, for later use, we summarize the results of these references. The eigenvalues of $X = CV^{11}, X_+ = CV^{12}, X_- = CV^{21}$ and T are given, respectively, by

$$\mu^{rs}(k) = \frac{1 - 2\,\delta_{r,s} + e^{\frac{\pi k}{2}}\,\delta_{r+1,s} + e^{-\frac{\pi k}{2}}\,\delta_{r,s+1}}{1 + 2\cosh\frac{\pi k}{2}} \tag{2.115}$$

$$t(k) = -e^{-\frac{\pi |k|}{2}} \tag{2.116}$$

where $-\infty < k < \infty$. The generating function for the eigenvectors is

$$f^{(k)}(z) = \sum_{n=1}^{\infty} v_n^{(k)} \frac{z^n}{\sqrt{n}} = \frac{1}{k}(1 - e^{-k \arctan z}) \tag{2.117}$$

The completeness and orthonormality equations for the eigenfunctions are as follows

$$\sum_{n=1}^{\infty} v_n^{(k)} v_n^{(k')} = \mathcal{N}(k)\delta(k-k'), \qquad \mathcal{N}(k) = \frac{2}{k}\sinh\frac{\pi k}{2}, \qquad \int_{-\infty}^{\infty} dk\, \frac{v_n^{(k)} v_m^{(k)}}{\mathcal{N}(k)} = \delta_{nm} \tag{2.118}$$

The spectrum of X is continuous and lies in the interval $[-1/3, 0)$. It is doubly degenerate except at $-\frac{1}{3}$. The continuous spectrum of X' lies in the same interval, but X' in addition has a discrete spectrum. To describe it we follow [50]. We consider the decomposition

$$M'^{rs} = \frac{1}{3}(1 + \alpha^{s-r} CU' + \alpha^{r-s} U'C) \tag{2.119}$$

where $\alpha = e^{\frac{2\pi i}{3}}$. It is convenient to express everything in terms of CU' eigenvalues and eigenvectors (see Appendix B). The discrete eigenvalues are denoted by ξ and $\bar{\xi}$. Since CU' is unitary they lie on the unit circle. They are more effectively represented via the parameter η, (B.1), which in turn is connected to the parameter b (B.3). To each value of b there corresponds a couple of values of η with opposite sign (except for $b = 0$ which implies $\eta = 0$).

The eigenvectors corresponding to the continuous spectrum are $V_N^{(k)}$ $(-\infty < k < \infty)$, while the eigenvectors of the discrete spectrum are denoted by $V_N^{(\xi)}$ and $V_N^{(\bar{\xi})}$. They form a complete basis. They will be normalized so that the completeness relation takes the form

$$\int_{-\infty}^{\infty} dk\, V_N^{(k)} V_M^{(k)} + V_N^{(\xi)} V_M^{(\xi)} + V_N^{(\bar{\xi})} V_M^{(\bar{\xi})} = \delta_{NM} \tag{2.120}$$

It has become familiar and very useful to expand all the relevant quantities in VSFT by means of this basis. To this end we define

$$a_k = \sum_{N=0}^{\infty} V_N^{(k)} a_N, \qquad a_\xi = \sum_{N=0}^{\infty} V_N^{(\xi)} a_N, \qquad a_{\bar{\xi}} = \sum_{N=0}^{\infty} V_N^{(\bar{\xi})} a_N$$

$$a_N = \int_{-\infty}^{\infty} dk\, V_N^{(k)}\, a_k + V_N^{(\xi)} a_\xi + V_N^{(\bar{\xi})} a_{\bar{\xi}} \tag{2.121}$$

and introduce the even and odd twist combinations

$$e_k = \frac{a_k + Ca_k}{\sqrt{2}}, \qquad e_\eta = \frac{a_\xi + Ca_\xi}{\sqrt{2}}, \qquad o_k = \frac{a_k - Ca_k}{i\sqrt{2}}, \qquad o_\eta = \frac{a_\xi - Ca_\xi}{i\sqrt{2}}, \quad (2.122)$$

The commutation relations among them are

$$[e_k, e^\dagger_{k'}] = \delta(k - k'), \qquad [e_\eta, e^\dagger_\eta] = 1, \qquad [o_k, o^\dagger_{k'}] = \delta(k - k'), \qquad [o_\eta, o^\dagger_\eta] = 1, \quad (2.123)$$

while all the other commutators vanish. The twist properties are defined by

$$Ca_k = a_{-k}, \qquad Ca_\xi = a_{\bar{\xi}},$$

2.4.2 Diagonalization of the twisted product

Knowing the fact that twisted Neumann coefficients can be easily symmetrized to take the form of (minus) the matter Neumann coefficients, we have for free the eigenvalues. However we would like to show, as a consistency check, that we can derive the twisted spectrum by purely conformal considerations, following the lines of [28] but now using the twisted conformal field theory of the ghost system. As we have seen before, the twist is done as

$$T'(z) = T(z) - \partial j_{gh}(z) \quad (2.124)$$

leading to

$$L'_n = L_n + n j_n + \delta_{n0} \quad (2.125)$$

where

$$\begin{aligned} L_n &= -\sum_{k=-\infty}^{\infty} (2n - k) : c_{n-k} b_k : \quad (2.126) \\ j_n &= \sum_{k=-\infty}^{\infty} : c_{n-k} b_k : \end{aligned}$$

To find the eigenvectors of Y we consider the $*'$ algebra derivation

$$K'_1 = L'_1 + L'_{-1} \quad (2.127)$$

and then we use the same formal arguments of [28]. The main difference here is that K'_1 acts on b and c oscillator in a different but complementary way, due to their (twisted) conformal properties

$$[K'_1, c_n] = -(n + 1)c_{n+1} - (n - 1)c_{n-1} \quad (2.128)$$

$$[K'_1, b_n] = -n\, b_{n+1} - n\, b_{n-1} \quad (2.129)$$

We can have c-type vectors v_n, as well as b-type vectors w_n, so K_1 has two different matrix representations. If we act on c oscillators we get

$$\begin{aligned} [K_1', v_n c_n] &= (K^{(c)} v) \cdot c \\ K^{(c)}_{nm} &= -(m+1)\delta_{n,m+1} - (m-1)\delta_{n,m-1} \end{aligned} \tag{2.130}$$

If we act on b oscillators we get

$$\begin{aligned} [K_1', w_n b_n] &= (K^{(b)} v) \cdot b - w_1 b_0 \\ K^{(b)}_{nm} &= -m\delta_{n,m+1} - m\delta_{n,m-1} \end{aligned} \tag{2.131}$$

These two matrices transpose to each other and obey

$$K^{(c)} = K^{(b)\,T} = A^{-1} K^{(b)} A \tag{2.132}$$

in particular they share eigenvalues. The matrix A is defined to be

$$A_{nm} = n\delta_{nm} \tag{2.133}$$

We shall begin by diagonalizing $K^{(c)}$ and determine its eigenvectors.

$$K^{(c)}\, v^k = k\, v^k \tag{2.134}$$

In order to do so we map this algebraic problem in a differential one, by defining the generating function

$$f_{v^k}(z) = \sum_{n=1}^{\infty} v^k_n z^n \tag{2.135}$$

so that

$$v^k_n = \oint_0 \frac{dz}{2\pi i} \frac{1}{z^{n+1}} f_{v^k}(z) \tag{2.136}$$

With trivial manipulations we find that (2.134) is equivalent to

$$\left(-(1+z^2)\frac{d}{dz} - (z - \frac{1}{z})\right) f_{v^k}(z) = k\, f_{v^k}(z) \tag{2.137}$$

which is easily integrated to give

$$f_{v^k}(z) = \frac{z}{z^2+1} \mathrm{e}^{-k\tan^{-1} z} \tag{2.138}$$

where we have chosen the overall normalization in order to $v^k_1 = 1$. As usual k is a continuous parameter spanning all the real axis.

To find the b-eigenvectors it is worth noting that $K^{(b)}$ is the same as in the matter case [28], so we simply get the result

$$f_{w^k}(z) = \frac{1}{k}(1 - \mathrm{e}^{-k\tan^{-1} z}) \tag{2.139}$$

As a consistency check note that due to (2.132) c-eigenvectors are related to b-eigenvectors by

$$v_n^k = n\, w_n^k \tag{2.140}$$

which in functional language reads

$$f_{v^k}(z) = z\frac{d}{dz} f_{w^k}(z) \tag{2.141}$$

It is trivial to check that this relation is identically satisfied.
Once the spectrum of K_1' is found, in order to find the spectrum of Y, we begin by considering the algebra of wedge states in the twisted CFT. A wedge state can be defined as

$$|N\rangle' = \left(|0\rangle'\right)_{*'}^{N-1} = \mathcal{N}_N' \exp\left(\sum_{n,m=1}^{\infty} c_n^\dagger\, (CT_N')_{nm}\, b_m^\dagger\right)|0\rangle' \tag{2.142}$$

These states satisfy the relation

$$|N+1\rangle' = |N\rangle' *' |0\rangle' \tag{2.143}$$

Following the same formal arguments of [72], we can write all T_N in function of the sliver matrix T[5]

$$T_N' = \frac{T' - T'^{N-1}}{1 - T'^N} \tag{2.144}$$

In particular we have

$$\begin{aligned} T_2' &= 0 & (2.145)\\ T_3' &= Y' & (2.146)\\ T_\infty' &= T' & (2.147)\end{aligned}$$

actually the last equation is well defined for $|T| \leq 1$, we will see a posteriori that the eigenvalues of T lie on the interval $(0,1]$.

Such wedge states can be defined as surface states in the twisted CFT [22]. Given a string field $|\phi\rangle = \phi'(0)|0\rangle'$ the wedge state $|N\rangle$ can be defined as[6]

$$'\langle N|\phi\rangle = \langle f_N \circ \phi(0)\rangle' \tag{2.148}$$

where the generating function of the surface state is given by

[5]Note the change of signs with respect to [72], they come out from the differences in the algebraic linear and non linear properties of the Neumann coefficients of the twisted CFT

[6]In the brackets insertion of the c_0 0 mode is intended, since all oscillators in the game start from the 1 component, we don't have any ambiguity

$$f_N(z) = \frac{N}{2}\tan\left(\frac{2}{N}\tan^{-1} z\right) \tag{2.149}$$

Now we consider the state $|2+\epsilon\rangle'$. This state can be given a representation in terms of the twisted Virasoro generators as [15]

$$\begin{aligned} |B\rangle' &= \exp(\epsilon V_-)\,|0\rangle' = |0\rangle' + \epsilon V'_-|0\rangle' + O(\epsilon^2) && (2.150)\\ V_- &= \sum_{n=1}^{\infty}(-1)^n \frac{1}{(2n-1)(2n+1)} L'_{-2n} && (2.151)\end{aligned}$$

Using the explicit form of the twisted Virasoro generators

$$L'_n = -\sum_{k=-\infty}^{\infty}(n-k) : c_{n-k} b_k : \tag{2.152}$$

we can find the relevant Neumann coefficients of the state $|2+\epsilon\rangle'$

$$|2+\epsilon\rangle' = \exp\left(\epsilon \sum_{n,m=1}^{\infty} c_n^\dagger\,(CB')_{nm}\,b_m^\dagger\right)|0\rangle' = |0\rangle' + \epsilon \sum_{n,m=1}^{\infty} c_n^\dagger\,(CB')_{nm}\,b_m^\dagger |0\rangle' + O(\epsilon^2) \tag{2.153}$$

the B'_{nm} coefficients can be computed by comparing (2.153) with (2.151), we get

$$B'_{nm} = \frac{1}{2}\left(1+(-1)^{n+m}\right)\frac{(-1)^{\frac{n-m}{2}} n}{(n+m)^2-1} \tag{2.154}$$

This coefficient is made diagonal with c-type eigenvectors

$$\sum_{m=1}^{\infty} B'_{nm} v_m^k = \sum_{m=1}^{\infty} B'_{nm} m\; w_m^k = \beta'(\kappa) v_n^k = \beta'(\kappa) n\; w_n^k \tag{2.155}$$

Now take $n=1$, all goes the same way as [28], except for a minus sign in the definition (2.154)

$$\beta'(k) = \frac{1}{2}\,\frac{\frac{\pi\kappa}{2}}{\sinh\frac{\pi\kappa}{2}} \tag{2.156}$$

From B'-eigenvalues we can find out the eigenvalues of the twisted sliver $\tau'(k)$, by inverting the relation (2.144) at $N = 2+\epsilon$

$$B' = \frac{T'\log(T')}{1-T'} \tag{2.157}$$

which is bijective in the range $T \in (0,1]$, in so doing we get

$$\tau'(k) = \mathrm{e}^{-\frac{\pi|\kappa|}{2}} \tag{2.158}$$

Then we can use the twisted wedge states formula at $N=3$ to get the eigenvalues of Y', which we call $y'(k)$

$$y'(k) = \frac{\tau'(k) - \tau'(k)^2}{1-\tau'(k)^3} = \frac{1}{2\cosh\frac{\pi\kappa}{2}+1} \tag{2.159}$$

To find the spectrum of the other two coefficients $Y'_{\pm}$ we use the relations

$$\begin{aligned} Y' + Y'_+ + Y'_- &= -1 \\ Y'_+ Y'_- &= Y'^2 + Y' \end{aligned}$$

solving them for $Y'_{\pm}$ we get

$$Y'_{\pm} = -\frac{1}{2}\left(1 + Y' \mp \sqrt{(1-3Y')(1+Y')}\right) = -\frac{1+\cosh\frac{\pi\kappa}{2} \pm \sinh\frac{\pi\kappa}{2}}{2\cosh\frac{\pi\kappa}{2}+1} \tag{2.160}$$

As expected they are exactly the opposite of the matter ones

2.4.3 Block diagonalization of non twisted star

Let's rewrite for the sake of clarity the general form of the matrices defining the usual ghost product

$$\mathcal{Y} = \begin{pmatrix} 1 & 0 \\ \vec{y} & Y \end{pmatrix} \tag{2.161}$$

$$\mathcal{Y}_{\pm} = \begin{pmatrix} 0 & 0 \\ \vec{y}_{\pm} & Y_{\pm} \end{pmatrix} \tag{2.162}$$

The $(0,0)$ component isolates one eigenvalue for each matrix

$$eig[\mathcal{Y}] = 1 \oplus eig[Y] \tag{2.163}$$

$$eig[\mathcal{Y}_{\pm}] = 0 \oplus eig[Y_{\pm}] \tag{2.164}$$

It is then straightforward to find the eigenvector relative to these eigenvalues, this is achieved by block diagonalizing such matrices

$$\hat{\mathcal{Y}} = \begin{pmatrix} 1 & 0 \\ 0 & Y \end{pmatrix} \tag{2.165}$$

$$\hat{\mathcal{Y}}_{\pm} = \begin{pmatrix} 0 & 0 \\ 0 & Y_{\pm} \end{pmatrix} \tag{2.166}$$

with the change of basis

$$\hat{\mathcal{Y}}_{(\pm)} = \mathcal{Z}^{-1}\mathcal{Y}_{(\pm)}\mathcal{Z} \tag{2.167}$$

$$\mathcal{Z} = \begin{pmatrix} 1 & 0 \\ \vec{f} & 1 \end{pmatrix} \tag{2.168}$$

$$\mathcal{Z}^{-1} = \begin{pmatrix} 1 & 0 \\ -\vec{f} & 1 \end{pmatrix} \tag{2.169}$$

where

$$\vec{f} = \frac{1}{1-Y}\vec{y} = -\frac{1}{Y_{\pm}}\vec{y}_{\pm} \tag{2.170}$$

The equality of the last expressions is a simple consequence of (2.88). Note that the eigenvector we find is the same which defines the kinetic operator Q as a c midpoint insertion [7] . Since the equation (2.170) has the solution (3.30), it might seem that $(Y, Y_\pm)$ (small matrices) cannot have the eigenvalues $(1,0)$, this is actually not true because (2.170) is not a relation in the full Hilbert space, but only in its twist-even subspace. As we will see the $(1,0)$ eigenvalues will have a corresponding one twist odd eigenvector, contrary with all the other eigenvalues which will have eigenvectors of both twist parity. The linear transformation (2.168) induces the following redefinition of the bc oscillators.

$$\tilde{c}_0 = c_0 + \sum_{n\geq 1} f_n (c_n + (-1)^n c_n^\dagger) = \mathcal{Q} \tag{2.171}$$

$$\tilde{c}_n = c_n \quad n \neq 0 \tag{2.172}$$

$$\tilde{b}_0 = b_0 \tag{2.173}$$

$$\tilde{b}_n = -f_n b_0 + b_n \quad n \neq 0 \tag{2.174}$$

where we have defined $(f_{-n} \equiv f_n)$. This is an equivalent representation of the bc system[8]

$$\{\tilde{b}_N, \tilde{c}_M\} = \delta_{N+M} \quad N, M = -\infty, ..., 0, ..., \infty \tag{2.175}$$

Block diagonalization of big matrices has then lead to the discovery of a twist even eigenvector with non vanishing 0–component. This eigenvector is not visible in Siegel gauge and, as we have seen, it corresponds to the midpoint of the (ghost part of the) string.

2.4.4 Diagonalization of the reduced product

Once we know the spectrum of the twisted product we can use the equality of the twisted sliver and the reduced sliver (i.e. sliver in Siegel gauge) to directly compute the spectrum of the reduced Neumann coefficients. Here again we can define "wedge"-states as

$$|N\rangle = \left(|\dot{0}\rangle\right)^{N-1}_{*_{b_0}} = \mathcal{N}_N \exp\left(\sum_{n,m=1}^{\infty} c_n^\dagger \, (CT_N)_{nm} \, b_m^\dagger\right) |\dot{0}\rangle \tag{2.176}$$

Which are defined by

$$|N+1\rangle = |N\rangle *_{b_0} |\dot{0}\rangle \tag{2.177}$$

The Neumann coefficients T_N, are given by[9]

$$T_N = \frac{T + (-T)^{N-1}}{1 - (-T)^N} \tag{2.178}$$

[7] This, from a different point of view, was also note in [25]

[8] In order to prove this, twist invariance of $\vec{f}$ is crucial $(C\vec{f} = \vec{f})$

[9] The expression is formally identical to the matter case, this is so because the linear and non linear properties of the reduced Neumann coefficients are isomorphic to the matter.

In particular we have

$$T_2 = 0 \tag{2.179}$$

$$T_3 = Y \tag{2.180}$$

$$T_\infty = T = T' \tag{2.181}$$

The last equality follows from the fact that the twisted sliver is identical to the reduced sliver in Siegel gauge. We recall here that the name “wedge states” is somehow misleading, since this states cannot be interpreted as surface state in the non twisted CFT, this is so because the star product in the usual CFT increase the ghost number (as opposed to the twisted star product). In this sense these states can be properly defined only algebraically via the reduced product.

At $N = 3$, we get the eigenvalues of Y, $y(k)$, from the eigenvalues of $T = T'$ (2.158)

$$y(k) = \frac{1}{2\cosh\frac{\pi\kappa}{2} - 1} \tag{2.182}$$

Using (2.88) we obtain the spectrum for the Neumann coefficients $Y_\pm$, which we call $y_\pm(k)$

$$y_\pm(k) = \frac{\cosh\frac{\pi\kappa}{2} \pm \sinh\frac{\pi\kappa}{2} - 1}{2\cosh\frac{\pi\kappa}{2} - 1} \tag{2.183}$$

Chapter 3

String Field Theory at the Tachyon Vacuum

3.1 The discovery of a non perturbative vacuum

Open String Field Theory is formulated around the D25-brane vacuum, exhibiting an instability due to the presence of the open string tachyon. As reviewed in the introduction, such an instability is understood as the instability of the D25–brane itself. Indeed bosonic D–branes (as well as D–branes anti D–branes pairs and non–BPS D–branes in superstring theory) do not possess any charge which can prevent them from decaying. To see if there is a stable point in the tachyon potential is a task that can be taken over by looking at the space time effective action of string field theory. Explicit numerical computations can be performed if the string spectrum is truncated up to a certain level, so as to have a finite number of spacetime fields. Level truncation is an approximation scheme by which one can recover a more and more precise effective field theory from the exact but somehow formal string field theory action. It consists in expanding the string field up to a certain level (the eigenvalue of the N operator) and in explicitly computing the action using the prescriptions of the previous section to compute $*$–products and *bpz*–inner products. A truncated string field takes the form

$$|\Psi\rangle = (\phi(x) + A_\mu(x)a_1^{\mu\dagger} + \ldots)c_1|0\rangle \tag{3.1}$$

Plugging this expression in the action one ends up with a local action for the component fields up to a certain level

$$\mathcal{S}(\Psi) = \int d^{26}xF(\varphi_i, \partial\varphi_i, ...) \tag{3.2}$$

This action is a purely spacetime action and one can extract form it an effective tachyon potential.

Here we do not attempt at all to give a review of the level truncation computations from which the tachyon potential has been obtained, we just quote that a strong evidence that a local minimum exists has been achieved (see [11] for a pedagogical review of the level truncation technique). By truncating the action at a finite level one ends up with an effective tachyon potential, that have the qualitative form showed in figure

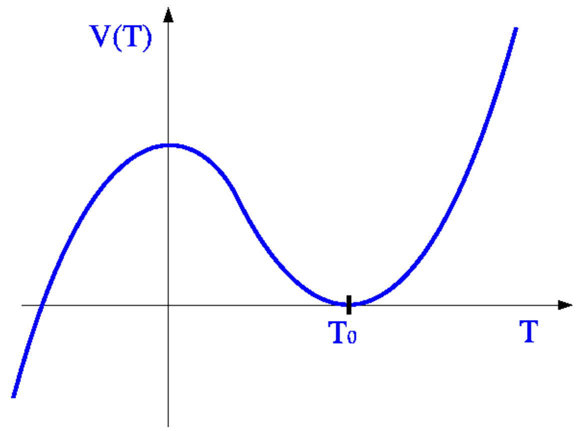

Figure 3.1: *The tachyon potential of open string field theory: the local maximum represents the unstable D25–brane, while the local minimum is the tachyon vacuum*

Lot of computations has been done to assure that the following statements about the tachyon vacuum are true

- The energy difference between the perturbative vacuum and the tachyon vacuum exactly matches with the D25–brane energy, hence it represents a configuration with no D–branes at all,
- The cohomology around the tachyon vacuum is trivial (at least at ghost number one), indicating that there are no physical perturbative open string states around it
- Lower dimensional D–branes can be obtained as tachyonic lumps in which, along the transverse directions, the tachyon reaches its minimum in the potential at $\pm\infty$ and it is vanishing at the origin, the energy of such lump solutions matches with the lower dimensional D–branes energy.

Such statements are known as Sen's conjectures and, thanks to the great amount of evidence reached, they are universally accepted as fundamental properties.

However it has been a challenge until now to exactly solve the string field theory equation of motion and to find the analytic form of the tachyon of the tachyon vacuum.

3.2 Vacuum String Field Theory

The remarkable properties of the tachyon vacuum suggest that Open String Field Theory should take its simplest form around it. As we have seen in chapter 2, when we expand

OSFT around a classical solution the action is reproduced up to a shift in the BRST operator, (1.15). So the only thing we need to write down OSFT at the tachyon vacuum is the new BRST operator which, in turn, is known if the classical solution representing the tachyon vacuum is known. Alternatively one can use Sen's conjectures to guess the form of the kinetic operator. In [18] a conjecture was put forward under the name of Vacuum String Field Theory. In this model the BRST operator is taken to be pure ghost: this is a particular implementation of the universality of the tachyon vacuum. In particular the proposed kinetic operator takes the form of a c-midpoint insertion, [22]

$$\mathcal{Q} = \frac{1}{2i}\left(c(i) - c(-i)\right) \tag{3.3}$$

Note that this operator is an eigenvector of the ghost product, see previous section. We recall that this operator has trivial cohomology due to the relation

$$\{\mathcal{Q}, b_0\} = 1 \quad \Rightarrow \quad \mathcal{Q}\psi = 0 \to \psi = \mathcal{Q}(b_0\psi) \tag{3.4}$$

Since both the star product and the kinetic operator are matter–ghost factorized, it is natural to search for solutions of the equation of motion which are matter/ghost factorized too. In particular, starting from the VSFT equation of motion

$$\mathcal{Q}\psi + \psi * \psi = 0 \tag{3.5}$$

and making the factorization ansatz

$$\psi = \psi_m \otimes \psi_{gh} \tag{3.6}$$

one ends up with the following equations, in the ghost and matter sector

$$\mathcal{Q}\psi_{gh} + \psi_{gh} *_{gh} \psi_{gh} \;=\; 0 \tag{3.7}$$

$$\psi_m *_m \psi_m \;=\; \psi_m \tag{3.8}$$

The equation in the matter sector are equations that defines idempotents (projectors) of the matter star algebra. The remaining chapters of this thesis are devoted to a detailed study of particular projectors that describe D–branes of any dimension and their decay towards the tachyon vacuum. On the other hand the ghost solution can be taken universally the same for any particular BCFT one wants to describe. In the next section we give a construction of the ghost solution by explicitly solving the ghost equation of motion, using the techniques learned in the previous chapter. Such solution will turn to have a divergent *bpz*–norm, this problem will be addressed in the next chapter, where both solutions in matter and ghost sector will be regularized by the dressing deformation. Such procedure will allow us to define the string coupling constant as an emergent dynamically generated quantity. For the time being we will just derive the simplest solution of the ghost equation of motion.

3.3 The universal ghost solution

We deal with the problem of finding a solution to (4.5)

$$\mathcal{Q}|\psi\rangle + |\psi\rangle * |\psi\rangle = 0 \tag{3.9}$$

We will do the task by working in the enlarged the Fock space of sec. (3.2.5). As the Hilbert space is enlarged $\mathcal{Q}$ must be modified, with respect to the conjectured form of the BRST operator (2.72) in VSFT, in the following way

$$\mathcal{Q} \to \mathcal{Q}_{(en)} = c_0 - \eta_0 + \eta_0^\dagger + \sum_{n=1}^{\infty} f_n(c_n + (-1)^n c_{-n}) \tag{3.10}$$

The first thing we would like to check is BRST invariance of the vertex, i.e.

$$\sum_{a=1}^{3} \mathcal{Q}^{(a)}_{(en)} |\tilde{V}_3\rangle_{(en)} = 0 \tag{3.11}$$

It is easy to verify that this equation is identically satisfied thanks to the first two eqs.(2.88), and thanks to addition of $-\eta_0$ in (3.10) ($\eta_0^\dagger$ passes through and annihilates the vacuum).

In order to solve equation (3.9) we proceed to find a solution to

$$|\hat{\psi}\rangle_3 = {}_1\langle\dot{\psi}|_2\langle\dot{\psi}|V_{123}\rangle \tag{3.12}$$

where $\hat{\psi}$ and $\dot{\psi}$ are the same state on the ghost number 2 and 1 vacuum, respectively. We choose the following ansatz

$$|\hat{\psi}\rangle = |\hat{S}_{(en)}\rangle = \mathcal{N}\exp\left(\sum_{n,m\geq 1} c_n^\dagger S_{nm} b_m^\dagger + \sum_{N\geq 0} c_N^\dagger S_{N0} \xi_0^\dagger\right)|\hat{0}\rangle \tag{3.13}$$

$$|\dot{\psi}\rangle = |\dot{S}_{(en)}\rangle = \mathcal{N}\exp\left(\sum_{n,m\geq 1} c_n^\dagger S_{nm} b_m^\dagger + \sum_{N\geq 0} c_N^\dagger S_{N0} \xi_0^\dagger\right)|\dot{0}\rangle \tag{3.14}$$

Following now the standard procedure, [18, 38], from (3.12), using (2.97), we get

$$\mathcal{T} = \mathcal{Y} + (\mathcal{Y}_+, \mathcal{Y}_-)\frac{1}{1-\Sigma\mathcal{V}}\Sigma\begin{pmatrix}\mathcal{Y}_- \\ \mathcal{Y}_+\end{pmatrix} \tag{3.15}$$

In RHS of these equations

$$\Sigma = \begin{pmatrix}\mathcal{T} & 0 \\ 0 & \mathcal{T}\end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix}\mathcal{Y} & \mathcal{Y}_+ \\ \mathcal{Y}_- & \mathcal{Y}\end{pmatrix}.$$

where $\mathcal{T} = C\mathcal{S}$ and $\mathcal{Y}, \mathcal{Y}_\pm$ have been defined by eq.(2.86).

We repeat once more that the matrix equation (3.15) is understood for 'large' matrices, which include the zeroth row and column, i.e. $\mathcal{Y} = \tilde{X}^{11} = C\tilde{N}^{11} = (1, \vec{y}, Y \equiv \tilde{X})$,

$\mathcal{T} = (1, \vec{t}, \tilde{T})$ and $\mathcal{S} = (1, \vec{s}, \tilde{S})$. This is a novelty of our treatment. In fact, solving eq.(3.15), we obtain the algebraic equation

$$\mathcal{T} = C\mathcal{S} = \frac{1}{2\mathcal{Y}}\Big(1 + \mathcal{Y} - \sqrt{(1-\mathcal{Y})(1+3\mathcal{Y})}\Big) \tag{3.16}$$

which splits into the relations

$$\begin{aligned} \mathcal{T}_{00} &= \mathcal{S}_{00} = 1 \\ \tilde{T} &= \frac{1}{2\tilde{X}}\Big(1 + \tilde{X} - \sqrt{(1-\tilde{X})(1+3\tilde{X})}\Big) \\ \vec{t} &= \frac{1-\tilde{T}}{1-\tilde{X}}\vec{y} \end{aligned} \tag{3.17}$$

The normalization constant $\mathcal{N}$ is, formally, given by

$$\mathcal{N} = \frac{1}{\det(1 - \Sigma\mathcal{V})} \tag{3.18}$$

However we notice that the (0,0) entry of $\Sigma\mathcal{V}$ is 1, so the determinant vanishes. Therefore we have to introduce a regulator $\varepsilon \to 0$, and write

$$\mathcal{N}_\varepsilon = \frac{1}{\varepsilon}\frac{1}{\det'(1 - \Sigma\mathcal{V})} \tag{3.19}$$

where $\det'$ is the determinant of the 'small' matrix part alone. This divergence is not present in the literature, [22, 26]. It is in fact related to the 1 eigenvalue of $\mathcal{T}$ and $\mathcal{Y}$ in the twist even sector (i.e. in the eigenspace of C with eigenvalue 1). This is therefore an additional divergence with respect to the usual one due to the 1 eigenvalue of $\tilde{X}$ in the twist–odd sector.

Now we prove that this solves (3.9). Indeed, after some elementary algebra, we arrive at the expression

$$\mathcal{Q}_{(en)}|\dot{S}\rangle + |\hat{S}\rangle = \Big(-c_n^\dagger\,[(\vec{s})_n - (C-\mathcal{S})_{nk}f_k] + c_0 - \eta_0\Big)|\dot{S}\rangle \tag{3.20}$$

We would like to find $\vec{f}$ so that the expression in square brackets in (3.20) vanishes. Using the last equation in (3.17) we see that this is true provided

$$\vec{y} = (1 - \tilde{X})\vec{f} \tag{3.21}$$

Now, by means of an explicit calculation, we verify that the solution to (3.21) is

$$f_n = \frac{1}{2}(1 + (-1)^n)(-1)^{\frac{n}{2}} \tag{3.22}$$

For inserting in the RHS of (3.21) both (3.22) and $\tilde{X}$ in the form

$$\tilde{X} = \frac{1}{3}(1 + C\tilde{U} + \tilde{U}C)$$

we see that the vanishing of f_n for n odd is consistent since $\vec{y}$ has no odd components, while for n even we have

$$y_{2n} = \sum_{k=1}^{\infty} \frac{2}{3}(-1)^k \Big(\delta_{2n,2k} - \tilde{U}_{2n,2k}\Big) \tag{3.23}$$

The second sum is evaluated with the use of the integral representation of $\mathcal{U}$ (2.59)

$$\begin{aligned}
\sum_{k=1}^{\infty}(-1)^k\,\tilde{U}_{2n,2k} &= \oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{2n+1}}\sum_{k=1}^{\infty}(-1)^k\frac{1}{w^{2k+1}}\frac{f(z)}{f(w)}\Big(\frac{1}{1+zw}-\frac{w}{w-z}\Big)\\
&= -\oint \frac{dz}{2\pi i}\oint \frac{dw}{2\pi i}\frac{1}{z^{2n+1}}\frac{1}{w}\frac{1}{1+w^2}\frac{f(z)}{f(w)}\Big(\frac{1}{1+zw}-\frac{w}{w-z}\Big)\\
&= -\oint \frac{dz}{2\pi i}\frac{1}{z^{2n+1}}f(z)\Big(1-\frac{1}{f(z)}\frac{1}{1+z^2}\Big)\\
&= -b_{2n}+\sum_{k=1}^{\infty}(-1)^k\delta_{2n,2k}
\end{aligned} \tag{3.24}$$

The δ-piece cancels with the one in (3.23), while the remaining one is precisely y_{2n}.

The derivation in (3.24) requires some comments. In passing from the first to the second line we use $\sum_{k=1}^{\infty}(-1)^k\frac{1}{w^{2k+1}} = -\frac{1}{w}\frac{1}{1+w^2}$, which converges for $|w|>1$. Therefore, in order to make sense of the operation, we have to move the w contour outside the circle of radius one. This we can do provided we introduce a regulator (see Appendix) to avoid the overlapping of the contour with the branch points of $f(w)$, which are located at $w=\pm i$. With the help of a regulator we move them far enough and eventually we will move them back to their original position. Now we can fully rely on the integrand in the second line of (3.24). Next we start moving the w contour back to its original position around the origin. In so doing we meet two poles (those referring to the $\frac{1}{1+w^2}$ factor), but it is easy to see that their contribution neatly vanishes due to the last factor in the integrand. The remaining contributions come from the poles at $w=z$ and at $w=0$. Their evaluation leads to the third line in (3.24). The rest is obvious.

As a result of this calculation we find that eq.(3.20) becomes

$$\mathcal{Q}_{(en)}|\dot{S}_{(en)}\rangle + |\hat{S}_{(en)}\rangle = (c_0-\eta_0)|\dot{S}_{(en)}\rangle \tag{3.25}$$

Finally, as a last step, we return to the original Fock space. A practical rule to do so is to drop all the double zero mode terms in the exponentials[1] (such as, for instance, $c_0\xi_0^\dagger$) and to impose the condition $c_0-\eta_0=0$ on the states, i.e. by considering all the states that differ by $c_0-\eta_0$ acting on some state as equivalent. The same has to be done also for $b_0-\xi_0^\dagger$ (paying attention not to apply both constraints simultaneously, because they do not commute). These rules are enough for our purposes. In this context the RHS of eq.(3.25) is in the same class as 0.

[1]Which is equivalent to normal ordering these terms. We thank A.Kling and S.Uhlmann for this suggestion.

Let us collect the results. In the original Fock space the three string vertex is defined by

$$\tilde{E}' = \sum_{n\geq 1, M\geq 0}^{\infty} c_n^{(a)\,\dagger} \tilde{V}_{nM}^{(ab)} b_M^{(b)\dagger} \tag{3.26}$$

eqs.(3.13,3.14) becomes

$$|\hat{S}\rangle = \mathcal{N}\exp\left(\sum_{n,m\geq 1} c_n^\dagger S_{nm} b_m^\dagger + \sum_{n\geq 1} c_n^\dagger S_{n0} b_0 \right) |\hat{0}\rangle \tag{3.27}$$

$$|\dot{S}\rangle = \mathcal{N}\exp\left(\sum_{n,m\geq 1} c_n^\dagger S_{nm} b_m^\dagger \right) |\dot{0}\rangle \tag{3.28}$$

It is now easy to prove, as a check, that

$$\mathcal{Q}|\dot{S}\rangle + |\hat{S}\rangle = 0 \tag{3.29}$$

where

$$\mathcal{Q} = c_0 + \sum_{n=1}^{\infty} (-1)^n (c_{2n} + c_{-2n}) \tag{3.30}$$

The above computation proves in a very direct way that the BRST operator is nothing but the midpoint insertion ($z = i$) of the operator $\frac{1}{2i}(c(z) - c(\bar{z}))$ [22]. A different proof of this identification, which makes use of the continuous basis of the $*$–algebra [58], was given in [26].

As an additional remark, we point out that the ghost action calculated in the enlarged and restricted Fock space are different, although they are both divergent due to the normalization (4.13).

In the next chapter we will use the Siegel gauge part of this solution and we will deform it in a particular way, so that (still being a solution) it can have a finite norm.

Chapter 4

Static solutions: D–branes

This chapter is devoted to find solutions of the matter projector equations that represent a single D–brane. We will use the operator formulation of String Field Theory given in chapter 1.

4.1 The matter sliver

To start with we recall some formulas relevant to VSFT. The action is

$$\mathcal{S}(\Psi) = -\frac{1}{g_0^2}\left(\frac{1}{2}\langle\Psi|\mathcal{Q}|\Psi\rangle + \frac{1}{3}\langle\Psi|\Psi * \Psi\rangle\right) \tag{4.1}$$

where

$$\mathcal{Q} = c_0 + \sum_{n>0}(-1)^n\left(c_{2n} + c_{-2n}\right) \tag{4.2}$$

Notice that the action (4.1) does not contain any singular normalization constant, as opposed to [58, 23]. This important issue will be discussed later, in connection with the emergence of the critical dimension $D = 26$. The equation of motion is

$$\mathcal{Q}\Psi = -\Psi * \Psi \tag{4.3}$$

and the ansatz for nonperturbative solutions is in the factorized form

$$\Psi = \Psi_m \otimes \Psi_g \tag{4.4}$$

where Ψ_g and Ψ_m depend purely on ghost and matter degrees of freedom, respectively. Then eq.(4.3) splits into

$$\mathcal{Q}\Psi_g = -\Psi_g *_g \Psi_g \tag{4.5}$$

$$\Psi_m = \Psi_m *_m \Psi_m \tag{4.6}$$

where $*_g$ and $*_m$ refers to the star product involving only the ghost and matter part. The action for this type of solution becomes

$$\mathcal{S}(\Psi) = -\frac{1}{6g_0^2}\langle\Psi_g|\mathcal{Q}|\Psi_g\rangle\langle\Psi_m|\Psi_m\rangle \tag{4.7}$$

$\langle\Psi_m|\Psi_m\rangle$ is the ordinary inner product, $\langle\Psi_m|$ being the *bpz* conjugate of $|\Psi_m\rangle$ (see below).

We have seen in the previous chapter how to find solutions to (4.5), this problem will be taken up again in section (4.5) of the present chapter. For the time being, as an introduction to the problem, let us concentrate on the matter part, eq.(4.6). We will mostly discuss solutions representing D25–branes which are translationally invariant, at the end of the chapter we will extend our construction to lower dimensional D–branes. As a consequence we set all the momenta to zero. So the integration over the momenta will be dropped and the only surviving part in E will be the one involving V^{ab}_{nm}. This is what we understand in the following by $*_m$, unless otherwise specified.

Let us now return to eq.(4.6). Its solutions are projectors of the $*_m$ algebra. We recall the simplest one, the sliver. It is defined by

$$|\Xi\rangle = \mathcal{N} e^{-\frac{1}{2}a^\dagger S a^\dagger}|0\rangle, \qquad a^\dagger S a^\dagger = \sum_{n,m=1}^{\infty} a_n^{\mu\dagger} S_{nm} a_m^{\nu\dagger}\eta_{\mu\nu} \tag{4.8}$$

This state satisfies eq.(4.6) provided the matrix S satisfies the equation

$$S = V^{11} + (V^{12}, V^{21})(1-\Sigma\mathcal{V})^{-1}\Sigma\begin{pmatrix} V^{21} \\ V^{12}\end{pmatrix} \tag{4.9}$$

where

$$\Sigma = \begin{pmatrix} S & 0 \\ 0 & S\end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} V^{11} & V^{12} \\ V^{21} & V^{22}\end{pmatrix}, \tag{4.10}$$

The proof of this fact is well–known, [54]. First one expresses eq.(4.10) in terms of the twisted matrices $X = CV^{11}, X_+ = CV^{12}$ and $X_- = CV^{21}$, together with $T = CS = SC$, where $C_{nm} = (-1)^n\delta_{nm}$. The matrices X, X_+, X_- are mutually commuting. Then, requiring that T commute with them as well, one can show that eq.(4.10) reduces to the algebraic equation

$$XT^2 - (1+X)T + X = 0 \tag{4.11}$$

The interesting solution is

$$T = \frac{1}{2X}(1 + X - \sqrt{(1+3X)(1-X)}) \tag{4.12}$$

The normalization constant $\mathcal{N}$ is calculated to be

$$\mathcal{N} = (\mathrm{Det}(1-\Sigma\mathcal{V}))^{\frac{D}{2}} \tag{4.13}$$

where D is the space time dimensionality. Note that, at this stage, we don't have any consistency reason to ask for critical dimension $D = 26$. The contribution of the sliver to the matter part of the action (see (4.7)) is given by

$$\langle\Xi|\Xi\rangle = \frac{\mathcal{N}^2}{(\det(1-S^2))^{\frac{D}{2}}} \tag{4.14}$$

Both eq.(4.13) and (4.14) are ill–defined and need to be regularized, after which they both turn out to vanish. This subject will be taken up again next, where we will introduce the dressing technique.

In Appendix A we collect a series of properties and results concerning the matrices X, X_-, X_+, T, together with other formulas that will be needed in the following.

4.2 Dressing the sliver

We have already pointed out that the sliver is a state with vanishing *bpz* norm. We want now to see if there are other projectors akin to it for which the norm can be given a non vanishing value.

The procedure we use is to deform the sliver Neumann coefficient S. To this end we first introduce the infinite vector $\xi = \{\xi_n\}$ which is chosen to satisfy the condition

$$\rho_1 \xi = 0, \qquad \rho_2 \xi = \xi, \tag{4.15}$$

Notice that this vector does not have any Lorentz label (compare with [30]). Next we set

$$\xi^T \frac{1}{1-T^2} \xi = 1, \qquad \xi^T \frac{T}{1-T^2} \xi = \kappa \tag{4.16}$$

where ^{T} denotes matrix transposition. Eqs.(4.16) will be studied in section 3. Our candidate for the *dressed sliver* solution is given by an ansatz similar to (4.8)

$$|\hat{\Xi}\rangle = \hat{\mathcal{N}} e^{-\frac{1}{2} a^\dagger \hat{S} a^\dagger} |0\rangle, \tag{4.17}$$

with S replaced by

$$\hat{S} = S + R, \qquad R_{nm} = \frac{1}{\kappa+1} \left(\xi_n (-1)^m \xi_m + \xi_m (-1)^n \xi_n\right) \tag{4.18}$$

As a consequence T is replaced by

$$\hat{T} = T + P, \qquad P_{nm} = \frac{1}{\kappa+1} \left(\xi_m \xi_n + \xi_n (-1)^{m+n} \xi_m\right) \tag{4.19}$$

From time to time a bra and ket notation will be used to represent P:

$$P = \frac{1}{\kappa+1} \left(|\xi\rangle\langle\xi| + |C\xi\rangle\langle C\xi|\right) \tag{4.20}$$

We require the dressed sliver to satisfy hermiticity, which amounts to imposing that the *bpz*–conjugate state coincide with the hermitian conjugate one. This in turn implies

$$|\xi\rangle\langle C\xi| + |\xi\rangle\langle C\xi| = |\xi^*\rangle\langle C\xi^*| + |\xi^*\rangle\langle C\xi^*|$$

We satisfy this condition by choosing ξ real. This means that κ is real (and negative). We remark at this point that the conditions (4.16) are not very stringent. The only thing one has to worry is that the lhs's are finite (this is the only true condition). Once this is true the rest follows from suitably rescaling ξ, so that the first equation is satisfied, and from the reality of ξ (see also next section).

We claim that $|\hat{\Xi}\rangle$ is a projector. The dressed sliver matrix $\hat{T}$ does not commute with X, X_-, X_+ (as T does), but we can nevertheless make use of the property $C\hat{T} = \hat{T}C$, because $CP = PC$. To prove our claim we must show that

$$V^{11} + (V^{12}, V^{21})(1 - \hat{\Sigma}\mathcal{V})^{-1}\Sigma \begin{pmatrix} V^{21} \\ V^{12} \end{pmatrix} = \hat{S} \tag{4.21}$$

where

$$\hat{\Sigma} = \begin{pmatrix} \hat{S} & 0 \\ 0 & \hat{S} \end{pmatrix} \tag{4.22}$$

We will in fact prove in detail that

$$X + (X_+, X_-)(1 - \hat{\mathcal{T}}\mathcal{M})^{-1}\hat{\mathcal{T}} \begin{pmatrix} X_- \\ X_+ \end{pmatrix} = \hat{T} \tag{4.23}$$

where

$$\hat{\mathcal{T}} = C\hat{\Sigma} = \mathcal{T} + \mathcal{P}, \qquad \mathcal{M} = C\mathcal{V}$$

To this end, let us define

$$\hat{\mathcal{K}} = 1 - \hat{\mathcal{T}}\mathcal{M} = 1 - \mathcal{T}\mathcal{M} - \mathcal{P}\mathcal{M} = \mathcal{K} - \mathcal{P}\mathcal{M} \tag{4.24}$$

The symbol $\mathcal{K}$ is the same as $\hat{\mathcal{K}}$ when the deformation P is absent, so it is the quantity relevant to the sliver. Now we write

$$\hat{\mathcal{K}}^{-1} = (1 - \hat{\mathcal{T}}\mathcal{M})^{-1} = \mathcal{K}^{-1}(1 - \mathcal{P}\mathcal{M}\mathcal{K}^{-1})^{-1}$$

We have

$$(1 - \mathcal{P}\mathcal{M}\mathcal{K}^{-1})^{-1}\mathcal{P} = \begin{pmatrix} 1 & \rho_1 - \kappa\rho_2 \\ \rho_2 - \kappa\rho_1 & 1 \end{pmatrix} \mathcal{P} \tag{4.25}$$

This can be shown either by expanding the lhs in power series or multiplying this equation from the left by $1 - \mathcal{P}\mathcal{M}\mathcal{K}^{-1}$ and verifying that it is an identity. To obtain this result one must use eq.(4.16) and the formulas in Appendix A, from which in particular one can derive

$$X_+\xi = X(T-1)\xi, \qquad X_-\xi = (1 - XT)\xi$$

Now we can evaluate the lhs of eq.(4.23)

$$\begin{aligned}
& X + (X_+, X_-)(1 - \hat{\mathcal{T}}\mathcal{M})^{-1}\hat{\mathcal{T}} \begin{pmatrix} X_- \\ X_+ \end{pmatrix} \\
&= X + (X_+, X_-)\mathcal{K}^{-1}(1 - \mathcal{P}\mathcal{M}\mathcal{K}^{-1})^{-1}(\mathcal{T} + \mathcal{P}) \begin{pmatrix} X_- \\ X_+ \end{pmatrix} \\
&= X + (\rho_1, \rho_2)\mathcal{T} \begin{pmatrix} X_- \\ X_+ \end{pmatrix} + (\rho_1, \rho_2)(1 - \mathcal{P}\mathcal{M}\mathcal{K}^{-1})^{-1}\mathcal{P} \begin{pmatrix} X_- \\ X_+ \end{pmatrix} \\
&\qquad + (\rho_1, \rho_2)(1 - \mathcal{P}\mathcal{M}\mathcal{K}^{-1})^{-1}\mathcal{P}\mathcal{M}\mathcal{K}^{-1}\mathcal{T} \begin{pmatrix} X_- \\ X_+ \end{pmatrix}
\end{aligned}$$

The first two terms in the rhs are exactly T. Next one notices that

$$(\rho_1, \rho_2)(1 - \mathcal{PMK}^{-1})^{-1} = (1, 1)$$

Therefore

$$\begin{aligned}
& X + (X_+, X_-)\hat{\mathcal{K}}^{-1}\hat{\mathcal{T}} \begin{pmatrix} X_- \\ X_+ \end{pmatrix} = T + PX_- + PX_+ + (1,1)P \begin{pmatrix} TX\rho_2 + TX_+\rho_1 \\ TX_-\rho_2 + TX\rho_1 \end{pmatrix} = \\
& = T + \frac{1}{\kappa+1}\Big(|C\xi\rangle\langle C\xi| X(T-1) + |\xi\rangle\langle\xi|(1-XT) + |C\xi\rangle\langle C\xi|(1-XT) + |\xi\rangle\langle\xi| X(T-1) \Big) \\
& \quad + \frac{1}{\kappa+1}\Big(|C\xi\rangle\langle C\xi| T(1-XT) + |\xi\rangle\langle\xi| XT + |C\xi\rangle\langle C\xi| XT + |\xi\rangle\langle\xi| T(1-XT) \Big) = \\
& = T + \frac{1}{\kappa+1}\left(|\xi\rangle\langle\xi| + |C\xi\rangle\langle C\xi| \right) = T + P = \hat{T}
\end{aligned} \tag{4.26}$$

In the passage to the last line we have used the identity $XT - X + T - XT^2 = 0$. This completes the proof that $\hat{\Xi}$ is a solution to (4.6).

We remark that, due to the arbitrariness of ξ, the result we have obtained brings into the game an infinite family of solutions to the equations of motion[1]. We shall see later that this result can be easily generalized. For the time being however we are interested in studying the properties of these new solutions.

The normalization constant $\hat{\mathcal{N}}$ is given by (see appendix C)

$$\hat{\mathcal{N}} = \mathrm{Det}(1 - \hat{\Sigma}\mathcal{V})^{\frac{D}{2}} = \mathrm{Det}(1 - \mathcal{TM})^{\frac{D}{2}} \mathrm{Det}(1 - \mathcal{PM}K^{-1})^{\frac{D}{2}} = \mathrm{Det}(1 - \mathcal{TM})^{\frac{D}{2}} \cdot \frac{1}{(\kappa+1)^D} \tag{4.27}$$

However, if one tries to compute the norm of this state (which corresponds to the its contribution to the action), i.e. $\langle\hat{\Xi}|\hat{\Xi}\rangle$, one finds an indeterminate result (as will be apparent from the calculation in section 6). It is evident that we have to introduce a regulator in order to end up with a finite action. Our idea is to introduce a numerical parameter ϵ in front of R in the definition of $\hat{\Xi}$. In this way we define new squeezed states $\hat{\Xi}_\epsilon$ characterized by the matrix $\hat{S}_\epsilon = S + \epsilon R$. But, before we come to that, a discussion of some issues concerning the vector ξ is in order.

4.3 A discussion on the half string vector ξ

In this section we will give a precise construction of the "half string" vector ξ. In so doing it is very convenient to use the continuous k basis of the star algebra.

In chapter 2 it was shown that the Neumann coefficients (X, X_+, X_-) can be simultaneously put in a continuous diagonal form as follows

$$X = \int_{-\infty}^{\infty} dk\, x(k)\, |k\rangle\langle k|, \qquad X_\pm = \int_{-\infty}^{\infty} dk\, x_\pm(k)\, |k\rangle\langle k| \tag{4.28}$$

[1]We believe this multiplicity of solutions to correspond mostly to gauge degrees of freedom.

The eigenvalues are given by[2]

$$
\begin{aligned}
x(k) &= -\frac{1}{1+2\cosh(\frac{\pi k}{2})} \\
x_{\pm}(k) &= \frac{1+\cosh(\frac{\pi k}{2}) \pm \sinh(\frac{\pi k}{2})}{1+2\cosh(\frac{\pi k}{2})}
\end{aligned}
$$

and the eigenvectors

$$
\begin{aligned}
|k\rangle &= \left(\frac{2}{k}\sinh(\frac{\pi k}{2})\right)^{-\frac{1}{2}} \sum_{n=1}^{\infty} v_n(k)|n\rangle \\
v_n(k) &= \sqrt{n}\oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{k}\left(1-e^{-k\tan^{-1}(z)}\right)
\end{aligned}
$$

These eigenvectors are normalized by the condition, [26],

$$
\langle k|k'\rangle = \delta(k-k')
$$

In this basis the sliver matrix T takes the remarkably simple form

$$
T = -\int_{-\infty}^{\infty} dk\, e^{-\frac{\pi|k|}{2}}\, |k\rangle\langle k|
$$

One should think at the real line spanned by k as a parametrization of the string itself in which the midpoint is represented by the $k=0$ eigenvector and the left (right) half by $k>0$ ($k<0$). This is easy to see once the form of the projectors ρ_1, ρ_2 is given in such a basis

$$
\begin{aligned}
\rho_1 &= \int_{-\infty}^{\infty} dk\,\theta(k)\,|k\rangle\langle k| = \int_{0}^{\infty} dk\,|k\rangle\langle k| \\
\rho_2 &= \int_{-\infty}^{\infty} dk\,\theta(-k)\,|k\rangle\langle k| = \int_{0}^{\infty} dk\,|-k\rangle\langle -k|
\end{aligned}
\tag{4.29}
$$

The value of these projectors at $k=0$ is a subtle point [24] and we will avoid this singular mode in the construction of the vector ξ. Since ξ is constrained by $\rho_2|\xi\rangle = |\xi\rangle$, $\rho_1|\xi\rangle = 0$, it is natural to parameterize it as

$$
|\xi\rangle = \int_{0}^{\infty} dk\,\xi(y)\,|-k\rangle \tag{4.30}
$$

where $y = \frac{\pi k}{2}$. Now the vector $|\xi\rangle$ is represented by the function $\xi(y)$, which has support on the positive real axis. The expressions (4.16) take the integral form

$$
\langle\xi|\frac{1}{1-T^2}|\xi\rangle = \frac{2}{\pi}\int_{0}^{\infty} dy\,\xi^2(y)\frac{1}{1-e^{-2y}} = 1 \tag{4.31}
$$

$$
\langle\xi|\frac{T}{1-T^2}|\xi\rangle = \frac{2}{\pi}\int_{0}^{\infty} dy\,\xi^2(y)\frac{-e^{-y}}{1-e^{-2y}} = \kappa \tag{4.32}
$$

[2] We hope the reader should not confuse k with κ.

Note that the denominator $1-T^2$ vanishes at $k=0$, so, in order to avoid infinities, we further require $\xi(y)$ to vanish rapidly enough at $y=0$. This means that the vector ξ does not excite the (zero momentum) midpoint mode. The space of functions with support on the positive axis, vanishing at the origin, and satisfying (4.31, 4.32) with finite κ, are spanned by a (numerable) infinite set of "orthogonal" functions defined by

$$\xi_n(y) = \left(\frac{\pi}{2}(1-e^{-2y})e^{-y}\right)^{\frac{1}{2}} L_n(y)\, y^q, \qquad q>0 \tag{4.33}$$

where $L_n(y)$ is the n-th Laguerre polynomial.
The normalization factor in front of the polynomials has been chosen in order to satisfy

$$\langle\xi_n|\frac{1}{1-T^2}|\xi_m\rangle = \lim_{q\to 0^+}\int_0^\infty dy\, e^{-y}y^{2q}L_n(y)L_m(y) = \delta_{nm} \tag{4.34}$$

In a similar fashion, using standard properties of Laguerre polynomials[3], one can prove that

$$\langle\xi_n|\frac{T}{1-T^2}|\xi_m\rangle = -\lim_{q\to 0^+}\int_0^\infty dy\, e^{-2y}y^{2q}L_n(y)L_m(y) = K_{nm} = -\frac{1}{2^{n+m}}\frac{(m+n)!}{n!m!} \tag{4.35}$$

A simple numerical analysis shows that the eigenvalues of the matrix K_{nm} lie in the interval $(-1,0)$. This is of course what one should expect once the normalization condition $\langle\xi\frac{1}{1-T^2}\xi\rangle = 1$ is imposed. In fact the condition (4.32) differs from (4.31) by the insertion of the matrix T, which has a spectrum covering (twice) the interval $(-1,0)$.
In order to prove that these half string vectors can be concretely defined as Fock space vectors, we shall see that it is possible to have a complete control on their norm as well, and that such norms are always positive . Using the same standard manipulations as before, we have

$$\begin{aligned}\langle\xi_n|\xi_m\rangle &= \langle\xi_n|\frac{1-T^2}{1-T^2}|\xi_m\rangle = \\ &= \lim_{q\to 0^+}\int_0^\infty dy\, e^{-y}(1-e^{-2y})y^{2q}L_n(y)L_m(y) = \delta_{nm} - \frac{2^{m-n}}{3^{n+m+1}}\sum_{p=0}^{n}4^p\binom{n}{p}\binom{m}{m-n+p}\end{aligned} \tag{4.36}$$

Again a simple numerical analysis shows that the eigenvalues of the matrix defined by the *rhs* of (4.36), lie in the interval $(0,1)$: this definitely ensures the existence of such vectors.
As we will see in the last section, we can build orthogonal projectors (in the sense of the star product and of the *bpz*-norm) by simply using different and orthogonal half-string vectors, where orthogonality is understood in the following sense

$$\langle\xi|\frac{1}{1-T^2}|\xi'\rangle = 0, \qquad \langle\xi|\frac{T}{1-T^2}|\xi'\rangle = 0, \tag{4.37}$$

---

[3] In particular we need the relation

$$L_n(\lambda y) = \sum_{p=0}^{n}\binom{n}{p}\lambda^{n-p}(1-\lambda)^p L_{n-p}(y)$$

In view of the above discussion it is obvious that one can always find a finite number of ξ_n's to construct any given number of mutually orthogonal vectors although the number of ξ_n's needed increases faster with respect to the number of orthogonal projectors.

4.4 The states $\hat{\Xi}_\epsilon$

After the digression of the previous section, let us return to the problem of regularizing the norm for the matter part of the dressed sliver. As anticipated in section 2, the (naive) definition (4.17) given in section 2 for the dressed sliver does not avoid ambiguities and indefiniteness, when we come to compute its norm. The determinants involved in such calculations are in general not well–defined. To evade this problem we deform the dressed sliver by introducing a parameter ϵ, so that we get the dressed sliver when $\epsilon = 1$. When $\epsilon \neq 1$ the state we obtain is, in general, not a $*$–algebra projector. We will define the dressed sliver as the limit of a sequence of such states.

Let us introduce the state

$$|\hat{\Xi}_\epsilon\rangle = \hat{\mathcal{N}}_\epsilon e^{-\frac{1}{2}a^\dagger \hat{S}_\epsilon a^\dagger}|0\rangle, \tag{4.38}$$

where

$$\hat{S}_\epsilon = S + \epsilon R, \tag{4.39}$$

As a consequence T is replaced by

$$\hat{T}_\epsilon = T + \epsilon P, \tag{4.40}$$

The states defined in this way are not in general projectors, but have very interesting properties. It is worth to make a short detour to illustrate them.

We would like to show that the states (4.38) define a continuous $*$–abelian 1–parameter family of states. First we show that they are closed under the $*$–product. Hence let us consider

$$|\hat{\Xi}_{\epsilon_1}\rangle * |\hat{\Xi}_{\epsilon_2}\rangle = \hat{\mathcal{N}}(\epsilon_1,\epsilon_2) e^{-\frac{1}{2}a^\dagger C(T+\epsilon_1 P)*(T+\epsilon_2 P)a^\dagger}|0\rangle \tag{4.41}$$

where we denote

$$(T+\epsilon_1 P)*(T+\epsilon_2 P) \equiv X + (X_+, X_-)(1-\hat{\mathcal{T}}_{\epsilon_1\epsilon_2}\mathcal{M})^{-1}\hat{\mathcal{T}}_{\epsilon_1\epsilon_2}\begin{pmatrix} X_- \\ X_+ \end{pmatrix} \tag{4.42}$$

and

$$\hat{\mathcal{T}}_{\epsilon_1\epsilon_2} \equiv \begin{pmatrix} \hat{T}_{\epsilon_1} & 0 \\ 0 & \hat{T}_{\epsilon_2} \end{pmatrix} \tag{4.43}$$

In order to compute this expression we need the generalized formula

$$(1-\mathcal{P}_{\epsilon_1\epsilon_2}\mathcal{M}\mathcal{K}^{-1})^{-1}\mathcal{P}_{\epsilon_1\epsilon_2} = \frac{1}{1+(1-\epsilon_1)(1-\epsilon_2)\kappa}\begin{pmatrix} (1-\epsilon_2)\kappa+1 & \epsilon_1(\rho_1-\kappa\rho_2) \\ \epsilon_2(\rho_2-\kappa\rho_1) & (1-\epsilon_1)\kappa+1 \end{pmatrix}\mathcal{P}_{\epsilon_1\epsilon_2} \tag{4.44}$$

One can prove this formula as an easy generalization of section 3. Alternatively one can check it directly by multiplying it on the left with $(1-\mathcal{P}_{\epsilon_1\epsilon_2}\mathcal{M}\mathcal{K}^{-1})$ (a detailed proof can be found in the appendix C). Then things are straightforward and we get

$$(T+\epsilon_1 P)*(T+\epsilon_2 P) = T+(\epsilon_1\star\epsilon_2)P \tag{4.45}$$

where we have defined the *abelian* multiplication law between real numbers

$$\epsilon_1\star\epsilon_2 = \frac{\epsilon_1\epsilon_2}{1+(1-\epsilon_1)(1-\epsilon_2)\kappa} \tag{4.46}$$

This product is easily shown to be associative

$$\begin{aligned}(\epsilon_1\star\epsilon_2)\star\epsilon_3 &= \epsilon_1\star(\epsilon_2\star\epsilon_3) = \\ &= \frac{\epsilon_1\epsilon_2\epsilon_3}{1+\kappa\,(2-\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_1\epsilon_2\epsilon_3+\kappa(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3))}\end{aligned} \tag{4.47}$$

and exhibits three idempotent elements

$$0\star 0=0, \qquad 1\star 1=1, \qquad \frac{\kappa+1}{\kappa}\star\frac{\kappa+1}{\kappa}=\frac{\kappa+1}{\kappa}$$

Note that 1 is the identity

$$\epsilon\star 1 = 1\star\epsilon = \epsilon$$

The inverse with respect to this product is given by

$$\epsilon^{\star-1} = \frac{(1-\epsilon)\kappa+1}{(1-\epsilon)\kappa+\epsilon} \tag{4.48}$$

so that

$$\epsilon\star\frac{(1-\epsilon)\kappa+1}{(1-\epsilon)\kappa+\epsilon}=1$$

We have two distinct null elements which are 0 and $\frac{\kappa+1}{\kappa}$

$$0\star\epsilon=\epsilon\star 0=0, \qquad \frac{\kappa+1}{\kappa}\star\epsilon=\epsilon\star\frac{\kappa+1}{\kappa}=\frac{\kappa+1}{\kappa}$$

The point $\{\infty\}$ is naturally in the domain as it can be reached from any $\epsilon\neq 0,1,\frac{\kappa+1}{\kappa}$ by $\star$–product

$$\epsilon\star\left(1+\frac{1}{(1-\epsilon)\kappa}\right)=\infty$$

The simultaneous presence of two null elements makes their product ambiguous

$$\frac{\kappa+1}{\kappa}\star 0 = \text{indeterminate}$$

Note in particular that we have

$$0^{\star-1}=\frac{\kappa+1}{\kappa}, \qquad \left(\frac{\kappa+1}{\kappa}\right)^{\star-1}=0$$

This is very reminiscent of what happens with real numbers when they are completed with ∞, in which case what is ambiguous is the product $0 \cdot \infty$. One should note actually that this is the same situation, deformed by the parameter κ, as in the limit $\kappa \to 0$ one recovers the usual product and, in particular, $\frac{\kappa+1}{\kappa} \to \infty$.

In view of the structure we have found, two new abelian subalgebra of the $*$–product are naturally identified. The first is $R \cup \{\infty\} \setminus \{\frac{\kappa+1}{\kappa}\}$ and contains, as projectors, the sliver ($\epsilon = 0$) and the dressed sliver ($\epsilon = 1$). The second is $R \cup \{\infty\} \setminus \{0\}$ and contains the projectors $\epsilon = 1$ and $\epsilon = \frac{\kappa+1}{\kappa}$. We will call the state identified by $\epsilon = \frac{\kappa+1}{\kappa}$ the *exotic* dressed sliver. Note also that these two algebras are isomorphic to each other via the inversion map (4.48).

Since we are dealing with projectors, normalization is needed. The normalization of all the states in the two algebras is completely fixed once we ask the sliver and the exotic sliver to be really null elements. A general element of the two algebras can be written as

$$|\hat{\Xi}_\epsilon\rangle^{(1,2)} = \mathcal{N}_\epsilon^{(1,2)} e^{-\frac{\epsilon x}{\kappa+1}} |\Xi\rangle \tag{4.49}$$

where $|\Xi\rangle$ is the usual sliver with its (vanishing) normalization and the superscript $^{(1,2)}$ labels the algebras, moreover we have identified

$$x = a_\mu^\dagger C(|\xi\rangle\langle\xi| + |C\xi\rangle\langle C\xi|) a_\nu^\dagger \eta^{\mu\nu}.$$

It is then easy to show that the star products of two such states is given by

$$|\hat{\Xi}_\epsilon\rangle^{(1,2)} * |\hat{\Xi}_\eta\rangle^{(1,2)} = \frac{\mathcal{N}_\epsilon^{(1,2)} \mathcal{N}_\eta^{(1,2)}}{\mathcal{N}_{\epsilon\star\eta}^{(1,2)}} \left(\frac{\kappa+1}{1+(1-\epsilon)(1-\eta)\kappa} \right)^D |\hat{\Xi}_{\epsilon\star\eta}\rangle^{(1,2)} \tag{4.50}$$

The second factor in the *rhs* comes from $\mathrm{Det}(1 - \hat{\mathcal{T}}_{\epsilon\eta}\mathcal{M})^{-\frac{1}{2}}$, see appendix C. In the first algebra the null element is the sliver ($\epsilon = 0$) and, of course, $\mathcal{N}_0^{(1)} = 1$ since the sliver is a projector by itself. The star product with another state of the same algebra is then

$$|\hat{\Xi}_0\rangle^{(1)} * |\hat{\Xi}_\epsilon\rangle^{(1)} = \mathcal{N}_\epsilon^{(1)} \left(\frac{\kappa+1}{1+(1-\epsilon)\kappa} \right)^D |\hat{\Xi}_0\rangle^{(1)}$$

which implies

$$\mathcal{N}_\epsilon^{(1)} = \left(\frac{1+(1-\epsilon)\kappa}{\kappa+1} \right)^D \tag{4.51}$$

With this choice of normalization we have, use eq.(4.46),

$$\frac{\mathcal{N}_\epsilon^{(1)} \mathcal{N}_\eta^{(1)}}{\mathcal{N}_{\epsilon\star\eta}^{(1)}} = \left(\frac{1+(1-\epsilon)(1-\eta)\kappa}{\kappa+1} \right)^D$$

so the first algebra closes with structure constant 1,

$$|\hat{\Xi}_\epsilon\rangle^{(1)} * |\hat{\Xi}_\eta\rangle^{(1)} = |\hat{\Xi}_{\epsilon\star\eta}\rangle^{(1)} \tag{4.52}$$

Note that the exotic sliver has, in this algebra, an extra vanishing normalization due to the dressing factor, (4.51), so it is naturally excluded. Concerning the inverse algebra one has first to note that, in order for the exotic sliver to be a projector it should be that

$$\mathcal{N}^{(2)}_{\frac{\kappa+1}{\kappa}} = \frac{1}{\kappa^D}$$

Now one should ask the exotic sliver to be a null element of the algebra

$$|\hat{\Xi}_{\frac{\kappa+1}{\kappa}}\rangle^{(2)} * |\hat{\Xi}_{\epsilon}\rangle^{(2)} = |\hat{\Xi}_{\frac{\kappa+1}{\kappa}}\rangle^{(2)}$$

which implies

$$\mathcal{N}^{(2)}_{\epsilon} = \left(\frac{\epsilon}{\kappa+1}\right)^D \tag{4.53}$$

In this case too the inverse algebra closes with structure constant 1,

$$|\hat{\Xi}_{\epsilon}\rangle^{(2)} * |\hat{\Xi}_{\eta}\rangle^{(2)} = |\hat{\Xi}_{\epsilon\star\eta}\rangle^{(2)} \tag{4.54}$$

Note that the dressed sliver has the same normalization and behaves as the identity in both algebras.

The next task is to compute the *bpz*–norm of such states; here we limit ourselves to a formal expression, since all of them are constructed on the sliver which is known to have vanishing norm. This formal expression will be suitably regularized in the next section.

Using results from the appendix C, for states belonging to the (1) algebra we obtain

$$^{(1)}\langle\hat{\Xi}_{\epsilon}|\hat{\Xi}_{\epsilon}\rangle^{(1)} = \left(\frac{(\mathcal{N}^{(1)}_{\epsilon})^2}{\det(1-\hat{T}^2_{\epsilon})^{\frac{1}{2}}}\mathrm{Det}(1-\mathcal{T}\mathcal{M})\right)^D \langle 0|0\rangle \tag{4.55}$$

$$= \frac{V^{(D)}}{(2\pi)^D}\left(\frac{(1+(1-\epsilon)\kappa)^2}{(1-\epsilon)(1+\kappa)(1+\kappa-\epsilon(\kappa-1))}\right)^D \left(\frac{\mathrm{Det}(1-\mathcal{T}\mathcal{M})}{(\det(1-T^2))^{\frac{1}{2}}}\right)^D \tag{4.56}$$

As we have just mentioned, this expression is formal, since, due to the fact that the third factor in the *rhs* is vanishing, all norms in this algebra vanish as well, except perhaps for $\epsilon = 1$ and $\epsilon = \frac{\kappa+1}{\kappa-1}$, for which the denominator of the second factor vanishes, and we get a $\frac{0}{0}$ expression.

A remark is in order concerning the state represented by $\epsilon = \frac{\kappa+1}{\kappa-1}$. This state is not a projector, but has the nice property of squaring to the dressed sliver, and can be therefore identified with a non trivial "square root" of unity

$$|\hat{\Xi}_{\frac{\kappa+1}{\kappa-1}}\rangle^{(1)} * |\hat{\Xi}_{\frac{\kappa+1}{\kappa-1}}\rangle^{(1)} = |\hat{\Xi}_{1}\rangle^{(1)}$$

It is quite natural therefore that, if the dressed sliver can have a finite norm, also its square root should. For what concerns the inverse algebra (2) all can be repeated with only slight modifications for the normalization factors $\mathcal{N}^{(2)}$, which never vanishes for states belonging to the algebra itself. Again we can have a non–vanishing norm for the dressed sliver and its square root, which has the same normalization as in the algebra (1). Therefore we will not repeat the computation of the norm. In any case, in the rest of the paper, we will deal only with the *first* algebra.

To end this section we would like to make a comment on the eigenvalues of the Neumann matrix of the dressed sliver, which hopefully clarifies some of the formulas used below. As we have remarked, this Neumann matrix does not commute with the sliver matrix T, so they cannot share their eigenvectors. However much can be said about the eigenvalues of $\hat{T}$. If the vector ξ is square–summable (as we suppose), P is a compact operator. Perturbing T by a compact operator does not modify its continuous spectrum, [81]. Therefore $\hat{T}$ must have the same continuous spectrum as T. In addition, however, it might have isolated eigenvalues of its own. It is easy to show that $\hat{T}$ does develop an extra discrete eigenvalue 1. This fact can be easily guessed from the result of appendix C

$$\det(1-\hat{T}_\epsilon) = (1-\epsilon)^2 \det(1-T), \tag{4.57}$$

which suggests that $\hat{T}$ has a doubly degenerate eigenvalue 1. It turns out that the corresponding eigenvectors have definite twist and are given by

$$|\chi_\pm\rangle = \frac{1}{1-T}(1\pm C)|\xi\rangle \tag{4.58}$$

as can be easily proved by applying (4.19) to the above expression.

This is in fact the reason why the *bpz* norm of the dressed sliver can be made finite by appropriately tuning the vanishing behavior induced by the midpoint $k=0$ and the divergent one induced by this discrete eigenvalue. We will see, in the study of the spectrum, that these new eigenvectors are responsible for creating an infinite tower of "descendants"of every physical state, with same mass and same polarization conditions as the initial state.

4.5 The dressed sliver action: matter part

In the previous section we have introduced a Fock space state, depending on a parameter ϵ, that interpolates between the sliver $\epsilon=0$ and the dressed sliver $\epsilon=1$. Now we intent to show that by its means, we can give a precise definition of the norm of the dressed sliver, so that both its norm and its action can be made finite.

As already mentioned above, the determinants in (4.13), (4.14) relevant to the sliver are ill–defined. They are actually well defined for any finite truncation of the matrix X to level L and need a regulator to account for its behaviour when $L\to\infty$. A regularization that fits particularly our needs was introduced by Okuyama [26] and we will use it here. It consists in using an asymptotic expression for the eigenvalue density $\rho(k)$ of X (see also section 3), $\rho(k)\sim\frac{1}{2\pi}\log L+\rho_{fin}(k)$, for large L, where $\rho_{fin}(k)$ is a finite contribution when $L\to\infty$, see [50]. This leads to asymptotic expressions for the various determinants we need. In particular the scale of L can be chosen in such a way that

$$\begin{aligned}
\det(1+T) &= h_+\, L^{-\frac{1}{3}} + \dots \\
\det(1-T) &= h_-\, L^{\frac{1}{6}} + \dots \\
\det(1-X) &= h_X\, L^{\frac{1}{9}} + \dots
\end{aligned} \tag{4.59}$$

where dots denote non–leading contribution when $L \to \infty$ and h_+, h_-, h_X are suitable numerical constants which arise due to the finite contribution in the eigenvalue density[4]. Our strategy consists in tuning L with ϵ in such a way as to obtain finite results. Let us start, as a warm up exercise, with the *bpz* norm $\langle \hat{\Xi}_\epsilon | \hat{\Xi}_\epsilon \rangle$. We have

$$\langle \hat{\Xi}_\epsilon | \hat{\Xi}_\epsilon \rangle = \frac{\hat{\mathcal{N}}_\epsilon^2}{[\det(1-\hat{S}_\epsilon^2)]^{D/2}} \langle 0||0\rangle \tag{4.60}$$

and (see previous section)

$$\hat{\mathcal{N}}_\epsilon = [\mathrm{Det}(1-\Sigma\mathcal{V})]^{D/2} \mathcal{N}_\epsilon^{(1)}, \qquad \mathcal{N}_\epsilon^{(1)} = \left(\frac{1+(1-\epsilon)\kappa}{\kappa+1}\right)^D \tag{4.61}$$

Likewise we have

$$\begin{aligned}
\det(1-\hat{S}_\epsilon^2) &= \det(1-\hat{T}_\epsilon^2) = \det(1-\hat{T}_\epsilon)\det(1+\hat{T}_\epsilon) \\
&= \det(1-T^2)\det(1-\epsilon P\frac{1}{1-T})\det(1+\epsilon P\frac{1}{1+T})
\end{aligned} \tag{4.62}$$

Using the results of appendix C we find

$$\det(1-\hat{S}_\epsilon^2) = \det(1-T^2)(1-\epsilon)^2 \left(\frac{\kappa+1-\epsilon(\kappa-1)}{\kappa+1}\right)^2 \tag{4.63}$$

Therefore, in the limit $\epsilon \to 1$ the dominant term will be

$$\det(1-\hat{S}_\epsilon^2) = \det(1-T^2)(1-\epsilon)^2 \left(\frac{2}{\kappa+1}\right)^2 \tag{4.64}$$

Now, recalling that $\mathrm{Det}(1-\Sigma\mathcal{V}) = \det(1-X)\det(1+T)$, and putting together all the above results, we find

$$\langle \hat{\Xi}_\epsilon | \hat{\Xi}_\epsilon \rangle = \left(h \frac{1}{4(\kappa+1)^2} \frac{L^{-\frac{5}{18}}}{(1-\epsilon)^2} + \dots \right)^{\frac{D}{2}} \langle 0|0\rangle, \qquad h = \frac{h_X^2 h_+}{h_-} \tag{4.65}$$

where dots denote irrelevant terms in the limit $\epsilon \to 1$ and $L \to \infty$. Therefore, if we assume that

$$1-\epsilon = sL^{-\frac{5}{36}} \tag{4.66}$$

[4]In particular, for any infinite matrix A which is diagonal in the k–basis, the determinant can be regularized by the level L as

$$\det(A) = h_A\, L^{\int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k)}, \qquad h_A = e^{\int_{-\infty}^{\infty} dk \rho_{fin}(k) A(k)}.$$

We thank D.Belov for a discussion on this point.

for some constant s, we have

$$\lim_{\epsilon\to 1}\langle\hat{\Xi}_\epsilon|\hat{\Xi}_\epsilon\rangle = \left(\frac{h}{4(\kappa+1)^2 s^2}\right)^{\frac{D}{2}}\langle 0|0\rangle \tag{4.67}$$

which may take any prescribed positive finite value [5] . The factor $\langle 0|0\rangle = \delta^{(D)}(0)$ is normalized to $\frac{V^{(D)}}{(2\pi)^D}$. We notice for later use that, in order for such prescription to be consistent, it must be that if we rescale $1-\epsilon$, $L^{-\frac{5}{36}}$ should be accordingly rescaled so that their ratio is always s. This is in order to guarantee that the limit be scale independent.

It would look natural to define the number (4.67) as the norm $\langle\hat{\Xi}|\hat{\Xi}\rangle$ of our *regularized dressed sliver*. However, as we shall see next, the regularization prescription defined by eqs.(4.65,4.66,4.67) does not guarantee that the equations of motion be satisfied in the action. In fact, as it turns out (see below),

$$\lim_{\epsilon\to 1}\langle\hat{\Xi}_\epsilon|\hat{\Xi}_\epsilon\rangle \neq \lim_{\epsilon\to 1}\langle\hat{\Xi}_\epsilon|\Xi_\epsilon * \hat{\Xi}_\epsilon\rangle \tag{4.68}$$

There is here a subtle problem. We delve into it by analyzing the quantity

$$\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}\rangle = \frac{\hat{\mathcal{N}}_{\epsilon_1}\hat{\mathcal{N}}_{\epsilon_2}}{\det(1-\hat{S}_{\epsilon_1}\hat{S}_{\epsilon_2})}\langle 0|0\rangle \tag{4.69}$$

The analysis carried out in Appendix C leads us to infinite many ways of taking the limit $\epsilon_1, \epsilon_2 \to 1$, with results that vary in a finite range. At one extreme we have the result obtained above, which corresponds to $\epsilon_1 = \epsilon_2 = \epsilon$. At the the other extreme we have the ordered limit

$$\lim_{\epsilon_1\to 1}\left(\lim_{\epsilon_2\to 1}\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}\rangle\right) \tag{4.70}$$

According to Appendix C, when ϵ_1 and ϵ_2 are in the vicinity of 1 we have

$$\begin{aligned}&\frac{1}{\langle 0|0\rangle}\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}\rangle\\ &= \left(\frac{\mathrm{Det}(1-\Sigma\mathcal{V})}{\sqrt{\det(1-S^2)}}\right)^D \left(\frac{1}{4(\kappa+1)^2}\right)^{\frac{D}{2}} \left(\frac{4}{(\kappa(1-\epsilon_1)(1-\epsilon_2)+1-\epsilon_1\epsilon_2)^2}\right)^{\frac{D}{2}} + \dots\end{aligned}$$

where dots denote non–leading terms. Now let as take the limit (4.70)

$$\begin{aligned}&\frac{1}{\langle 0|0\rangle}\lim_{\epsilon_1\to 1}\left(\lim_{\epsilon_2\to 1}\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}\rangle\right) \qquad (4.71)\\ &= \lim_{\epsilon_1\to 1}\left(\frac{\mathrm{Det}(1-\Sigma\mathcal{V})}{\sqrt{\det(1-S^2)}}\right)^D \left(\frac{1}{4(\kappa+1)^2}\right)^{\frac{D}{2}} \left(\frac{4}{(1-\epsilon_1)^2}\right)^{\frac{D}{2}} + \dots\\ &= \lim_{\epsilon_1\to 1}\left(\frac{h}{(\kappa+1)^2}\right)^{\frac{D}{2}} \left(\frac{L^{-\frac{5}{36}}}{1-\epsilon_1}\right)^D + \dots = \left(\frac{h}{(\kappa+1)^2 s_1^2}\right)^{\frac{D}{2}}\end{aligned}$$

[5]It is obvious that the constants $\kappa+1$ and h could be absorbed in s.

provided

$$1-\epsilon_1 = s_1 L^{-\frac{5}{36}} \tag{4.72}$$

It is easy to see that if we reverse the order of the limits in (4.70) we obtain the same result.

Between this result and (4.67) there is a discrepancy, a factor of 4. This factor can be absorbed into a redefinition of s, $s_1 = 2s$. Had we adopted still another method of taking the limit we would have obtained a result in between. In conclusion there are infinite many ways of deriving the norm starting from $\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}\rangle$, they all lead to the same result up to a redefinition of the s factor.

Now the question is: do we have a criterion to select among all these different limits? The answer is: yes, we do. It is the requirement that the equation of motion be satisfied, i.e. we must have

$$\lim_{\epsilon_1,\epsilon_2\to 1}\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}\rangle = \lim_{\epsilon_1,\epsilon_2,\epsilon_3\to 1}\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}*\hat{\Xi}_{\epsilon_3}\rangle \tag{4.73}$$

The analysis carried out in Appendix C of the expression in the *rhs* tells us that once again there are infinite many ways to calculate the triple limit, and there are infinite many ways to satisfy (4.73). For instance, the limit $\epsilon_1=\epsilon_2=\epsilon_3\to 1$ does not satisfy (4.73), while the criterion of the ordered limits does, i.e. that

$$\lim_{\epsilon_1\to 1}\left(\lim_{\epsilon_2\to 1}\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}\rangle\right) = \lim_{\epsilon_1\to 1}\left(\lim_{\epsilon_2\to 1}\left(\lim_{\epsilon_3\to 1}\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}*\hat{\Xi}_{\epsilon_3}\rangle\right)\right) \tag{4.74}$$

First we notice that due to the symmetry of $\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}*\hat{\Xi}_{\epsilon_3}\rangle$ (see Appendix C), the order $1,2,3$ in the last limit is irrelevant. What is relevant is that the limits are taken in succession. Now, using the formulas of the previous section and of Appendix C, it is easy to see that

$$\lim_{\epsilon_3\to 1}\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}*\hat{\Xi}_{\epsilon_3}\rangle = \lim_{\epsilon_3\to 1}\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2\star\epsilon_3}\rangle = \langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}\rangle \tag{4.75}$$

Therefore (4.73) follows.

As we mentioned before, there are other ways of taking the limit $\epsilon_i\to 1$ while satisfying (4.73). However the ordered limits seem to have a privileged status, as we will try to explain next. We would like to show that the equation of motion (4.6) holds in a more general sense than eq.(4.74). In other words we would like that

$$\lim_{\epsilon_1\to 1}\langle\Psi|\hat{\Xi}_{\epsilon_1}\rangle = \lim_{\epsilon_1,\epsilon_2\to 1}\langle\Psi|\hat{\Xi}_{\epsilon_1}*\hat{\Xi}_{\epsilon_2}\rangle \tag{4.76}$$

for 'any' state Ψ. In order to appreciate the problem one should recall that the limiting procedure is necessary whenever evaluations of determinants are involved, otherwise it is irrelevant. Therefore if Ψ is any state of the Fock space constructed by applying to the vacuum a polynomial of the string creation operators, eq.(4.76) holds; the only proviso is that, since $\hat{\Xi}_\epsilon$ contains a normalization which vanishes when $L\to\infty$ (but it diverges in the ghost case, see below), we must take this limit as the last operation.

The validity of eq.(4.76) may be in danger only when Ψ is a close relative to $\hat{\Xi}$. We have already seen how to deal with the case $\hat{\Xi}_\epsilon$. The conclusion does not change if the $\hat{\Xi}_\epsilon$ is multiplied by a polynomial of the string creation operators or even by a coherent state constructed out of the latter. One may ask what happens when Ψ coincides with $\hat{\Xi}$ itself. In this case the expressions under the limit symbols in eq.(4.76) make sense, and we have to make sure that the equation holds. It is easy to see that, once again, it holds with the ordered limiting procedure. The set of states Ψ for which (4.76) holds, does not exhaust all the states one can think of, however it contains all Fock space states as well as all the states that are relevant in our discussion. To characterize these limitations we say that the EOM holds in a weak sense.

From now on we assume the ordered limit procedure as the good limiting procedure. In particular the norm of $\hat{\Xi}$ is defined by eq.(4.71).

What we have achieved so far is to prove that it is possible to assign a finite positive number to the expression (norm) $\langle\hat{\Xi}|\hat{\Xi}\rangle$, in a way which is consistent with the matter equation of motion. It does not mean that a state exists in the Hilbert space which is the limit of $\hat{\Xi}_\epsilon$ when $\epsilon \to 1$. In order to show this one would have to prove that the number $||\hat{\Xi}_{\epsilon_1} - \hat{\Xi}_{\epsilon_2}||^2$ becomes smaller and smaller when $\epsilon_1, \epsilon_2 \to 1$. In such a case Hilbert space completeness would guarantee the existence of a limiting state. Now

$$||\hat{\Xi}_{\epsilon_1} - \hat{\Xi}_{\epsilon_2}||^2 = \langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_1}\rangle + \langle\hat{\Xi}_{\epsilon_2}|\hat{\Xi}_{\epsilon_2}\rangle - 2\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}\rangle \tag{4.77}$$

The first two terms are similar to (4.60), while the last term has been calculated in (C.39). From the latter equation it is evident that the Cauchy condition would be satisfied if the term in the expression in the second line of eq.(C.39) were to approach 1 when $\epsilon_1, \epsilon_2 \to 1$. However, as seen in Appendix C1, this quantity remains at a finite distance from 1 when $\epsilon_1, \epsilon_2 \to 1$, unless one takes $\epsilon_1 = \epsilon_2$. The conclusion is that we cannot satisfy the Cauchy condition for the sequence $\hat{\Xi}_\epsilon$.

Therefore, while the regularization procedure defined above guarantees that we can associate a positive finite number to the symbol $\langle\hat{\Xi}|\hat{\Xi}\rangle$, it does not allow us to associate any Hilbert space state to $\hat{\Xi}$. *The state $\hat{\Xi}$ lives outside the Hilbert space.* A careful treatment of this problem would require embedding the string theory Hilbert space into a larger space with suitably defined topology, according to which $\lim_{\epsilon\to 1} \hat{\Xi}_\epsilon = \hat{\Xi}$ makes full sense. This interesting issue goes beyond the scope of the present dissertation.

4.6 The ghost dressed sliver

In this section our purpose is to find the ghost companion of the regularized dressed sliver solution discussed above. The previous analysis for the matter part can be easily extended also to the ghost part.

Let us start with the definition of the $*_g$ product:

$$|\widetilde{\Psi}\rangle *_g |\widetilde{\Phi}\rangle \;=\; {}_1\langle\widetilde{\Psi}|\, {}_2\langle\widetilde{\Phi}|\widetilde{V}_3\rangle \tag{4.78}$$

where the ghost part of the 3-strings vertex is defined by

$$|\widetilde{V}_3\rangle = \exp\left[\sum_{r,s=1}^{3}\left(\sum_{n,m=1}^{\infty} c_n^{(r)\dagger}\widetilde{V}_{nm}^{rs} b_m^{(s)\dagger} + \sum_{n=1}^{\infty} c_n^{(r)\dagger}\widetilde{V}_{n0}^{rs} b_0^{(s)}\right)\right]\prod_{r=1}^{3}\left(c_0^{(r)}c_1^{(r)}\right)|0\rangle_{123} \tag{4.79}$$

Here $c_n^{(r)}$ and $b_n^{(r)}$ are the standard ghost oscillator modes of the r-th string, which satisfy

$$\left\{b_n^{(r)}, c_m^{(s)\dagger}\right\} = \delta_{nm}\delta_{rs}\ , \qquad b_n^{(r)\dagger} = b_{-n}^{(r)}\ , \quad c_n^{(r)\dagger} = c_{-n}^{(r)}$$

and $|0\rangle_{123} \equiv |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$ is the tensor product of the $\mathrm{SL}(2,\mathbf{R})$-invariant ghost vacuum states, normalized such that

$$\langle 0|c_1^\dagger c_0 c_1|0\rangle = 1\ .$$

The symbols $\widetilde{V}_{nm}^{rs}$ and $\widetilde{V}_{n0}^{rs}$ are coefficients computed in [53, 71, 1, 16] and their properties necessary for us here are listed in appendix A. The *bpz* conjugation properties are

$$bpz\left(c_n^{(r)}\right) = (-1)^{n+1}c_{-n}^{(r)}\ , \qquad bpz\left(b_n^{(r)}\right) = (-1)^{n}b_{-n}^{(r)}\ .$$

It was shown in [23] that there is a simple solution of the ghost field equation (4.5) in the form of the squeezed state

$$|\widetilde{\Xi}\rangle = \widetilde{\mathcal{N}}\exp\left(\sum_{n,m=1}^{\infty} c_n^\dagger \widetilde{S}_{nm} b_m^\dagger\right) c_1|0\rangle\ , \tag{4.80}$$

where the matrix $\widetilde{S}$ satisfies the equation

$$\widetilde{S} = \widetilde{V}^{11} + (\widetilde{V}^{12}, \widetilde{V}^{21})(I - \widetilde{\Sigma}\widetilde{\mathcal{V}})^{-1}\widetilde{\Sigma}\begin{pmatrix}\widetilde{V}^{21}\\ \widetilde{V}^{12}\end{pmatrix}\ , \tag{4.81}$$

which has exactly the same form as (4.9) ($\widetilde{\Sigma}$ and $\widetilde{\mathcal{V}}$ are defined as in (4.10)), but now with tildes. As before one introduces $\widetilde{X} = C\widetilde{V}^{11}$, $\widetilde{X}_+ = C\widetilde{V}^{12}$ and $\widetilde{X}_- = C\widetilde{V}^{21}$ (see appendix A for properties). As the $\widetilde{X}_i$'s satisfy the same algebraic relations as the X_i's, one can construct solutions to (4.81) the same way as for the matter part. The solution we are interested in, in terms of $\widetilde{T} = C\widetilde{S}$, is

$$\widetilde{T} = \frac{1}{2\widetilde{X}}\left(1 + \widetilde{X} - \sqrt{(1+3\widetilde{X})(1-\widetilde{X})}\right)$$

The normalization constant is

$$\widetilde{\mathcal{N}} = -\left[\mathrm{Det}(1 - \widetilde{\Sigma}\widetilde{\mathcal{V}})\right]^{-1}\ . \tag{4.82}$$

The contribution of the ghost part to the action is given by

$$\langle\widetilde{\Xi}|\mathcal{Q}|\widetilde{\Xi}\rangle = \widetilde{\mathcal{N}}^2 \det(1-\widetilde{S}^2) \tag{4.83}$$

Now the determinants in eqs. (4.82) and (4.83) are both vanishing, in such a way that the ghost part of the action diverges (see below). When one combines this with the results for the matter part (using (4.4), (4.13) and (4.14)) one finds [26] that both normalization constant and action of the overall state vanish.

Following the analysis of the matter part, we consider deformations of this solution. We introduce two real vectors $\beta = \{\beta_n\}$ and $\delta = \{\delta_n\}$ which satisfy

$$\widetilde{\rho}_1\beta = \widetilde{\rho}_1\delta = 0, \qquad \widetilde{\rho}_2\beta = \beta, \qquad \widetilde{\rho}_2\delta = \delta\ . \tag{4.84}$$

We also set

$$\langle\beta|\frac{1}{1-\widetilde{T}^2}|\delta\rangle = 1\ , \qquad \langle\beta|\frac{\widetilde{T}}{1-\widetilde{T}^2}|\delta\rangle = \widetilde{\kappa} \tag{4.85}$$

where the first equation fixes the relative normalization of β and δ, and the second defines $\widetilde{\kappa}$. Note that one can repeat the analysis of section 3: since the eigenvalues of the ghost sliver matrix $\widetilde{T}$ are the opposite of the eigenvalues of the corresponding matter matrix T, it follows that $\widetilde{\kappa}$ is non–negative.

We now dress the ghost part of the sliver and introduce the squeezed state

$$|\widehat{\widetilde{\Xi}}_{\tilde{\epsilon}}\rangle = \widehat{\widetilde{\mathcal{N}}}_{\tilde{\epsilon}}\, e^{c^\dagger \widehat{\widetilde{S}}_{\tilde{\epsilon}} b^\dagger} c_1|0\rangle \tag{4.86}$$

where instead of $\widetilde{S}$ we now have

$$\widehat{\widetilde{S}}_{\tilde{\epsilon}} = \widetilde{S} + \tilde{\epsilon}\widetilde{R}\ , \qquad \widetilde{R} = \frac{1}{\widetilde{\kappa}+1}(|C\delta\rangle\langle\beta| + |\delta\rangle\langle C\beta|)$$

It it easy to see that $\widehat{\widetilde{S}}^*_{\tilde{\epsilon}} = C\widehat{\widetilde{S}}_{\tilde{\epsilon}}C$ for β, δ real, which means that the string field is real.

Let us now calculate $\widehat{\widetilde{\Xi}}_{\tilde{\epsilon}} *_g \widehat{\widetilde{\Xi}}_{\tilde{\eta}}$, where both states have the same β and δ. If one defines the reduced $*_{b_0}$–product as in chapter 2,

$$\widehat{\widetilde{\Xi}}_{\tilde{\epsilon}} *_{b_0} \widehat{\widetilde{\Xi}}_{\tilde{\eta}} \equiv b_0\left(\widehat{\widetilde{\Xi}}_{\tilde{\epsilon}} *_g \widehat{\widetilde{\Xi}}_{\tilde{\eta}}\right)\ , \tag{4.87}$$

then one can immediately see that it can be calculated using the vertex (4.79) but *without* terms containing b_0 modes (reduced vertex). Then the calculation of the reduced product (4.87) repeats essentially the calculation in the matter sector of sec. 4, the only differences being that untilded objects are replaced by the corresponding tilded ones and, more important, the determinants are raised to the power -2/D with respect to the corresponding matter ones (this is because of the anticommutativity of ghosts). The result is then

$$\begin{aligned} |\widehat{\widetilde{\Xi}}_{\tilde{\epsilon}}\rangle *_{b_0} |\widehat{\widetilde{\Xi}}_{\tilde{\eta}}\rangle &= \frac{\widehat{\widetilde{\mathcal{N}}}_{\tilde{\epsilon}}\widehat{\widetilde{\mathcal{N}}}_{\tilde{\eta}}}{\widehat{\widetilde{\mathcal{N}}}_{\tilde{\epsilon}\star\tilde{\eta}}} \mathrm{Det}(1-\widetilde{\Sigma}_{\tilde{\epsilon}\tilde{\eta}}\widetilde{\mathcal{V}})\,|\widehat{\widetilde{\Xi}}_{\tilde{\epsilon}\star\tilde{\eta}}\rangle \\ &= \frac{\widehat{\widetilde{\mathcal{N}}}_{\tilde{\epsilon}}\widehat{\widetilde{\mathcal{N}}}_{\tilde{\eta}}}{\widehat{\widetilde{\mathcal{N}}}_{\tilde{\epsilon}\star\tilde{\eta}}}\left[\frac{1+(1-\tilde{\epsilon})(1-\tilde{\eta})\widetilde{\kappa}}{\widetilde{\kappa}+1}\right]^2 \mathrm{Det}(1-\widetilde{\Sigma}\widetilde{\mathcal{V}})\,|\widehat{\widetilde{\Xi}}_{\tilde{\epsilon}\star\tilde{\eta}}\rangle\ , \end{aligned} \tag{4.88}$$

where the $\star$ multiplication rule is defined in (4.46).

Now, it was shown in [25] that, if states A and B are in the subspace spanned by coherent states, then the $*_g$ product can be obtained from the reduced product using

$$A *_g B = \mathcal{Q}\left(A *_{b_0} B\right) \ , \tag{4.89}$$

which applied to (4.88) gives

$$|\hat{\hat{\Xi}}_{\tilde{\epsilon}}\rangle *_g |\hat{\hat{\Xi}}_{\tilde{\eta}}\rangle = \frac{\hat{\hat{\mathcal{N}}}_{\tilde{\epsilon}}\hat{\hat{\mathcal{N}}}_{\tilde{\eta}}}{\hat{\hat{\mathcal{N}}}_{\tilde{\epsilon}\star\tilde{\eta}}}\left[\frac{1+(1-\tilde{\epsilon})(1-\tilde{\eta})\widetilde{\kappa}}{\widetilde{\kappa}+1}\right]^2 \mathrm{Det}(1-\widetilde{\Sigma}\widetilde{\mathcal{V}})\mathcal{Q}|\hat{\hat{\Xi}}_{\tilde{\epsilon}\star\tilde{\eta}}\rangle \ . \tag{4.90}$$

However, a more careful derivation is needed because states like (4.86) are (at least apparently) not of the required form and it could be risky to use the above argument. In appendix C we give a direct proof that (4.90) is correct.

At this point one should observe a formal similarity between eq. (4.90) and the corresponding one in the matter sector (4.50). In fact, one can now basically repeat the arguments of sections 4 and 5 with only minor modifications.

First, it is natural to choose a normalization such that (4.90) has the following form

$$|\hat{\hat{\Xi}}_{\tilde{\epsilon}}\rangle *_g |\hat{\hat{\Xi}}_{\tilde{\eta}}\rangle = -\mathcal{Q}|\hat{\hat{\Xi}}_{\tilde{\epsilon}\star\tilde{\eta}}\rangle \tag{4.91}$$

Again, there are two different normalizations with this property, given by

$$\hat{\hat{\mathcal{N}}}^{(1)}_{\tilde{\epsilon}} = -\left(\frac{\widetilde{\kappa}+1}{1+(1-\tilde{\epsilon})\widetilde{\kappa}}\right)^2\left[\mathrm{Det}(1-\widetilde{\Sigma}\widetilde{\mathcal{V}})\right]^{-1} \tag{4.92}$$

$$\hat{\hat{\mathcal{N}}}^{(2)}_{\tilde{\epsilon}} = -\left(\frac{\widetilde{\kappa}+1}{\tilde{\epsilon}}\right)^2\left[\mathrm{Det}(1-\widetilde{\Sigma}\widetilde{\mathcal{V}})\right]^{-1} \ . \tag{4.93}$$

The first one is singular in $\tilde{\epsilon}=\frac{\tilde{\kappa}+1}{\tilde{\kappa}}$, and the second one in $\tilde{\epsilon}=0$. From now on we use exclusively the first one, and drop the $^{(1)}$ in superscript.

From (4.91) it follows that our states (4.86) satisfy the ghost equation of motion when

$$\tilde{\epsilon}\star\tilde{\epsilon}=\tilde{\epsilon} \tag{4.94}$$

and we already know that it is true only for

$$\tilde{\epsilon}=0,\ 1,\ \frac{\widetilde{\kappa}+1}{\widetilde{\kappa}} \tag{4.95}$$

Again, beside the Hata-Kawano solution (i.e. the solution with $\tilde{\epsilon}=0$), we obtain in addition two families of solutions, depending on the choice of β and δ.

Now we show that for the solution with $\tilde{\epsilon}\to 1$ (ghost part of the dressed sliver) we can define a finite action. We shall consider first the kinetic term, for which we need

$$\begin{aligned}\langle\hat{\hat{\Xi}}_{\tilde{\epsilon}_1}|\mathcal{Q}\,|\hat{\hat{\Xi}}_{\tilde{\epsilon}_2}\rangle &= \langle\hat{\hat{\Xi}}_{\tilde{\epsilon}_1}|c_0|\hat{\hat{\Xi}}_{\tilde{\epsilon}_2}\rangle = \hat{\hat{\mathcal{N}}}_{\tilde{\epsilon}_1}\hat{\hat{\mathcal{N}}}_{\tilde{\epsilon}_2}\det(1-\hat{\hat{S}}_{\tilde{\epsilon}_1}\hat{\hat{S}}_{\tilde{\epsilon}_2})\\ &= \left(1-\prod_{i=1}^{2}\frac{\tilde{\epsilon}_i}{1+(1-\tilde{\epsilon}_i)\widetilde{\kappa}}\right)^2\frac{\det(1-\hat{\hat{S}}^2)}{\left[\mathrm{Det}(1-\widetilde{\Sigma}\widetilde{\mathcal{V}})\right]^2} \ .\end{aligned} \tag{4.96}$$

where in the last line we used (4.92) and (C.35). It was shown in [26] that the level truncation regularization at the leading order leads to

$$\frac{\det(1-\widehat{\widetilde{S}}^2)}{\left[\mathrm{Det}(1-\widetilde{\Sigma}\widetilde{\mathcal{V}})\right]^2} = \frac{L^{\frac{11}{18}}}{\tilde{h}} + \dots \tag{4.97}$$

where $\tilde{h}$ is the numerical factor analogous to h for the ghost part. The *rhs* of (4.97) diverges when the cutoff is lifted, i.e., when $L \to \infty$. But, as for the matter part, we see that if we let $\tilde{\epsilon} \to 1$ in a specific way, the expression (4.96) can be made finite. Following our discussion from sec. 5 we use the ordered limits procedure, which, using (4.97), gives

$$\begin{aligned}
\lim_{\tilde{\epsilon}_1\to 1}\left(\lim_{\tilde{\epsilon}_2\to 1}\langle\widehat{\widetilde{\Xi}}_{\tilde{\epsilon}_1}|\mathcal{Q}\,|\widehat{\widetilde{\Xi}}_{\tilde{\epsilon}_2}\rangle\right) &= \lim_{\tilde{\epsilon}_1\to 1}\left[1-\frac{\tilde{\epsilon}_1}{1+(1-\tilde{\epsilon}_1)\tilde{\kappa}}\right]^2 \frac{\det(1-\widehat{\widetilde{S}}^2)}{\left[\mathrm{Det}(1-\widetilde{\Sigma}\widetilde{\mathcal{V}})\right]^2} \\
&= \lim_{\tilde{\epsilon}_1\to 1}\left[\frac{(\tilde{\kappa}+1)(1-\tilde{\epsilon}_1)}{1+(1-\tilde{\epsilon}_1)\tilde{\kappa}}\right]^2 \frac{L^{\frac{11}{18}}}{\tilde{h}} + \dots\,.
\end{aligned} \tag{4.98}$$

Therefore, if we assume that

$$1-\tilde{\epsilon}_1 = \tilde{s}L^{-\frac{11}{36}} \tag{4.99}$$

for some constant $\tilde{s}$, we have

$$\lim_{\tilde{\epsilon}_1\to 1}\left(\lim_{\tilde{\epsilon}_2\to 1}\langle\widehat{\widetilde{\Xi}}_{\tilde{\epsilon}_1}|\mathcal{Q}\,|\widehat{\widetilde{\Xi}}_{\tilde{\epsilon}_2}\rangle\right) = (\tilde{\kappa}+1)^2\,\frac{\tilde{s}^2}{\tilde{h}}\,. \tag{4.100}$$

which defines a finite value for the kinetic term in the action.

The calculation for the cubic term in the action goes along similar lines. One obtains here too that the ordered limits preserve the equation of motion,

$$\lim_{\tilde{\epsilon}_1\to 1}\left(\lim_{\tilde{\epsilon}_2\to 1}\left(\lim_{\tilde{\epsilon}_3\to 1}\langle\widehat{\widetilde{\Xi}}_{\tilde{\epsilon}_1}|\widehat{\widetilde{\Xi}}_{\tilde{\epsilon}_2} *_g \widehat{\widetilde{\Xi}}_{\tilde{\epsilon}_3}\rangle\right)\right) = -\lim_{\tilde{\epsilon}_1\to 1}\left(\lim_{\tilde{\epsilon}_2\to 1}\langle\widehat{\widetilde{\Xi}}_{\tilde{\epsilon}_1}|\mathcal{Q}\,|\widehat{\widetilde{\Xi}}_{\tilde{\epsilon}_2}\rangle\right) \tag{4.101}$$

It is worth noting that, using the results of chapter 2, the ghost companion of the dressed sliver can be easily shown to be (proportional to) a projector of the bc–twisted $*$–product.

4.6.1 Overall regularized action

Now we are ready to draw the conclusion concerning the regularized action. We collect the results (4.71,4.100) and plug them into (4.7). The action of the regularized dressed sliver is

$$-\frac{\mathcal{S}(\hat{\Psi})}{V^{(D)}} = \frac{1}{6g_0^2(2\pi)^D}\,\frac{(\tilde{\kappa}+1)^2\tilde{s}^2}{(\kappa+1)^D s^D}\,\frac{h^{\frac{D}{2}}}{\tilde{h}} \tag{4.102}$$

The value of the *rhs* can now be tuned to the physical value of the D25–brane tension. We stress that, apart from g_0, the parameters in the *rhs* are not present in the initial action, but arise from the regularization procedure[6]. More comments on this point can be found in section 8.

We would also like to point out that the regularized action (4.102) is not the only possibility. We could, for instance, connect in various ways the ghost and matter asymptotic expansions, to get an overall finite action. We could perhaps use also the limits $\kappa, \tilde{\kappa} \to -1$. At this stage we cannot decide what the best prescription is. Hopefully the study of the spectrum will shed light on this problem.

4.7 Other finite norm solutions

In this section we discuss a few further issues concerning dressed slivers, without going into detailed calculations.

• **Multiply dressed slivers.** The most obvious generalization of the dressed sliver definition (4.17) consists in adding to $\hat{S}$ another operator R' with the same structure as R and ξ replaced by ξ', with

$$\rho_1 \xi' = 0, \qquad \rho_2 \xi' = \xi', \tag{4.103}$$

and

$$\xi'^T \frac{1}{1-T^2}\xi' = 1, \qquad \xi'^T \frac{T}{1-T^2}\xi' = \kappa' \tag{4.104}$$

the components of ξ' being real and κ' a real number. The matrix $\hat{T}$ will be replaced by

$$\hat{T}' = T + P + P', \qquad P' = \frac{1}{\kappa'+1}\left(|\xi'\rangle\langle\xi'| + |C\xi'\rangle\langle C\xi'|\right) \tag{4.105}$$

The obvious question is whether this new state is a projector. In general it is not, but if ξ' satisfies the 'orthogonality' conditions

$$\xi^T \frac{1}{1-T^2}\xi' = 0, \qquad \xi^T \frac{T}{1-T^2}\xi' = 0 \tag{4.106}$$

then it is easy to repeat the proof of section 2 and conclude that the squeezed state with structure matrix $\hat{S}' = S + R + R'$ is in fact a projector. On the basis of section 3, one can see that the conditions (4.106) are easy to implement.

Again, the norm (and the action) of this new projector is ill–defined. We can introduce deformation parameters ϵ before P and ϵ' before P', and repeat what we did in section 3, 4 and 5. For instance, for ϵ, ϵ' near 1, denoting by $\hat{S}_{\epsilon,\epsilon'}$ the relevant Neumann matrix,

$$\det(1 - \hat{S}^2_{\epsilon,\epsilon'}) = \det(1-T^2)(1-\epsilon)^2(1-\epsilon')^2 \frac{16}{(\kappa+1)^2(\kappa'+1)^2} \tag{4.107}$$

$$\mathrm{Det}(1 - \hat{\Sigma}'_{\epsilon,\epsilon'}\mathcal{V}) = \mathrm{Det}(1 - \mathcal{T}\mathcal{M}) \frac{1}{(\kappa+1)^2(\kappa'+1)^2} \tag{4.108}$$

[6]We remark that κ, $\tilde{\kappa}$, $h, \tilde{h}$ could be reabsorbed in the free parameters s,$\tilde{s}$.

and so on. It is obvious that we can add to $\hat{S}$ as many perturbations as we wish and still get projectors. For instance, if we add R'', specified by ξ'', with the same properties as ξ, the only condition we have to impose is that ξ'' be orthogonal to both ξ and ξ' in the sense of eq.(4.106).

• **Other projectors.** Starting from the dressed sliver solutions it is rather easy to construct many others which are $*$–orthogonal to the dressed sliver, according to the construction initiated in [30] and fully implemented in [67]. First we introduce a real vector $\zeta^\mu = \{\zeta_n^\mu\}$ (notice the Lorentz index!), which is chosen to satisfy the conditions

$$\rho_1 \zeta^\mu = 0, \qquad \rho_2 \zeta^\mu = \zeta^\mu, \qquad \forall \mu \tag{4.109}$$

and

$$\langle \zeta^\mu | \frac{1}{1-T^2} | \zeta^\nu \rangle \eta_{\mu\nu} = 1, \qquad \langle \zeta^\mu | \frac{T}{1-T^2} | \zeta^\nu \rangle \eta_{\mu\nu} = \lambda \tag{4.110}$$

Next we set

$$\mathbf{x} = -(a^{\mu\dagger} \zeta^\nu \, \eta_{\mu\nu})\, (a^{\mu\dagger} C \zeta^\nu \eta_{\mu\nu}), \tag{4.111}$$

introduce the Laguerre polynomials $L_n(\mathbf{x}/\lambda)$ and define the states $|\hat{\Lambda}_n\rangle$ as follows

$$|\hat{\Lambda}_n\rangle = (-\lambda)^n L_n\Big(\frac{\mathbf{x}}{\lambda}\Big) |\hat{\Xi}\rangle \tag{4.112}$$

where λ is an arbitrary real constant, and $|\hat{\Xi}\rangle$ is the dressed sliver.

If, in addition to the above conditions, ζ^μ are 'orthogonal' to the dressing vector ξ,

$$\langle \zeta^\mu | \frac{1}{1-T^2} | \xi \rangle = 0, \qquad \langle \zeta^\mu | \frac{T}{1-T^2} | \xi \rangle = 0, \qquad \text{for any} \quad \mu, \tag{4.113}$$

it is not hard to generalize the proofs of [30],[67] and conclude that

$$|\hat{\Lambda}_n\rangle * |\hat{\Lambda}_m\rangle = \delta_{n,m} |\hat{\Lambda}_n\rangle \tag{4.114}$$

$$\langle \hat{\Lambda}_n | \hat{\Lambda}_m \rangle = \delta_{n,m} \langle \hat{\Xi} | \hat{\Xi} \rangle \tag{4.115}$$

As explained in section 3, the additional conditions (4.113) are easy to comply with.

• **Lump solutions.** In VSFT lump solutions of any dimension have been found, [54, 30]. They are candidates to represent lower dimensional branes. By definition, they are not translational invariant in a subset of directions (the transverse ones). In order to find such solutions we cannot drop anymore the momentum dependence in the transverse directions. We therefore proceeds switch to the oscillator representation of the zero modes, given in chapter 2.

Since all the calculations we have done throughout the present chapter depend uniquely on such properties, we can repeat everything almost *verbatim*. So, there will be a matrix

T' given by a formula (4.12), with a normalization (4.13) and a *bpz* norm (4.14), where all the entries are primed. Next we introduce the dressed sliver exactly as before. To this end first we define the infinite vector $\xi' = \{\xi'_N\}$ satisfying the condition

$$\rho'_1 \xi' = 0, \qquad \rho'_2 \xi' = \xi', \tag{4.116}$$

and

$$\xi'^T \frac{1}{1-T'^2} \xi' = 1, \qquad \xi'^T \frac{T'}{1-T'^2} \xi' = \kappa \tag{4.117}$$

where κ is the same number as in (4.16). The transverse dressed sliver is defined by

$$|\hat{\Xi}_\perp\rangle = \hat{\mathcal{N}}' e^{-\frac{1}{2} a^\dagger \hat{S}' a^\dagger} |\Omega_b\rangle \tag{4.118}$$

where

$$\hat{S}' = S' + R', \qquad R'_{MN} = \frac{1}{\kappa+1}\left(\xi'_M (-1)^N \xi'_N + \xi'_N (-1)^M \xi'_M\right) \tag{4.119}$$

and so on. The proofs of section 3 can be repeated, given the diagonal structure of Neumann matrices with zero modes [79]. Once again we introduce a deformation parameter ϵ (the same as in section 4!) and repeat the derivations of section 5 (where D, for the transverse directions, equals $k-1$).

One of the most remarkable results of VSFT is the reproduction of the ratio of tensions for brane of different dimensions. It is important to verify that our regularization procedure does not alter this ratio.

It is easy to show that in the present case the ratio of tensions for brane of adjacent dimensions can be written as follows

$$\frac{\mathcal{T}_{24-k}}{2\pi \mathcal{T}_{25-k}} = \frac{3}{\sqrt{2\pi b^3}} \left(V_{00} + \frac{b}{2}\right)^2 \frac{(\det(1-X')^{3/4} \det(1+3X')^{1/4}}{(\det(1-X)^{3/4} \det(1+3X)^{1/4}} \cdot f(\epsilon,\kappa) \tag{4.120}$$

The factor $f(\kappa,\epsilon)$ is due to dressing. However it is elementary to prove that this factor is actually 1. What remains is the same as in [54]. It was proven numerically [54] and analytically [80] that the ratio at the *rhs* of (4.120) is exactly 1, thus reproducing the expected ratio.

It goes without saying that one can easily introduce a constant background B field in the transverse directions, along the lines of [66, 63].

4.8 Role of the critical dimension

In this section we would like to comment about the emergence of the critical dimension in our procedure and, more generally in VSFT. Let us start from the normalized action

$$S[\hat{\psi}] = -\frac{1}{g_0^2}\left(\frac{1}{2}\langle\hat{\psi}|\mathcal{Q}|\hat{\psi}\rangle + \frac{1}{3}\langle\hat{\psi}|\hat{\psi} * \hat{\psi}\rangle\right) \tag{4.121}$$

By means of the operator field redefinition [84]

$$\psi = e^{-\frac{1}{4}\ln\gamma(K_2-4)}\hat{\psi} \tag{4.122}$$

it can be brought to the form

$$S'[\psi] = -\frac{1}{g_0^2\gamma^3}\left(\frac{1}{2}\langle\psi|\mathcal{Q}|\psi\rangle + \frac{1}{3}\langle\psi|\psi * \psi\rangle\right) = -\frac{1}{g_0^2}\left(\frac{1}{2\gamma}\langle\tilde{\psi}|\mathcal{Q}|\tilde{\psi}\rangle + \frac{1}{3}\langle\tilde{\psi}|\tilde{\psi} * \tilde{\psi}\rangle\right) \tag{4.123}$$

where $\tilde{\psi} = \gamma\psi$. Both forms of the action have been considered previously in the literature, [58, 23], in the limit $\gamma \to 0$, implying a singular normalization of the action. What we have shown above is that free effective parameters appear in the process of regularizing the classical action so that a singular normalization of the latter can be avoided. This remark is of more consequence than it looks at first sight. The point is that the redefinition (4.122) can harmlessly be implemented only in $D = 26$. In noncritical dimensions, as a consequence of such a redefinition, an anomaly appears, [52]. In the course of our derivation above the critical dimension has never featured, but this remark brings it back into the game. This has an important consequence: setting $\gamma = g_0^{2/3}$ in the middle term of eq.(4.123), it is evident that in critical dimensions we can make any parameter to completely disappear from the action by means of a field redefinition. So, in $D = 26$, the value of the brane tension is dynamically produced and not put in by hand.

The very reason for this is that the family of operators $K_n = L_n - (-1)^n L_{-n}$ leaves the action cubic term invariant (only in D=26) while it acts linearly on the kinetic term as, [84]

$$[K_{2n}, \mathcal{Q}] = -4n(-1)^n\mathcal{Q} \tag{4.124}$$

In other words $\mathcal{Q}$ is an "eigenvector" of K_{2n}, and so every parameter can be absorbed by a field redefinition. In OSFT, on the other hand, one cannot implement a redefinition like (4.122) since Q_B does not transform as an eigenvector of K_{2n}, so the coupling constant there is really a free parameter in the action.

Let us elaborate more on this aspect. We remark that both the string fields ψ and $\hat{\psi}$ above satisfy the same EOM. Therefore there seems to exist different solutions of the EOM corresponding to the same energy, and, on the other hand, a given solution can be attributed different tensions (depending on what constant we put in front of the action, which does not affect the EOM). Since any constant put in front of the action in VSFT in critical dimension can be absorbed via a field redefinition, it is illusory to try to cure this problem by multiplying the action by some constant. This is a fact of the leading pure ghost form of VSFT in critical dimension and we have to come to terms with it (if we don't want to give up matter/ghost factorization). It is apparent from the above that VSFT in its leading matter ghost factorized form does not predict the exact value of the D-brane tension, but rather makes room for it to emerge dynamically. It is at this point that dressing comes handy. We have showed that in the theory there naturally arise scaling constants s and $\tilde{s}$ (see eq. (6.25) there) that can be adjusted to the physical value of the D-brane tension. Therefore the answer to the above puzzle is that if we redefine the string field in the action, the parameters s and $\tilde{s}$ should be scaled accordingly in such a way as to preserve the physical value of the brane tension. Of course, in this way, we are left with a multiplicity of solutions corresponding to the same tension which are gauge equivalent.

Chapter 5

Open strings states

In the previous chapter we have explicitly constructed a solution representing a D25–brane. Its tension is produced dynamically via a regularization scheme (dressing) that is consistent only in the critical dimension $D = 26$. This chapter is devoted to analyze the small fluctuations of this solution. We will see that on shell–fluctuations are in one–to–one correspondence with open string states on a D25–brane, hence they correspond to marginal boundary deformations of the BCFT representing the D25–brane.

5.1 The linearized equation of motion

Let us call for simplicity $\Phi_0 = |\hat{\Xi}\rangle \otimes |\hat{\widetilde{\Xi}}\rangle$ the overall (matter+ghost) solution we have just studied in the previous chapter. If we write $\Psi = \Phi_0 + \phi$, the action becomes

$$\mathcal{S}(\Psi) = \mathcal{S}(\Phi_0) - \frac{1}{g_0^2}\left(\frac{1}{2}\langle\phi|\mathcal{Q}_0|\phi\rangle + \frac{1}{3}\langle\phi|\phi * \phi\rangle\right) \tag{5.1}$$

where

$$\mathcal{Q}_0\phi = \mathcal{Q}\phi + \Phi_0 * \phi + \phi * \Phi_0 \tag{5.2}$$

The equation of motion for small fluctuations around the solution Φ_0 is therefore

$$\mathcal{Q}\phi + \Phi_0 * \phi + \phi * \Phi_0 = 0 \tag{5.3}$$

The solutions to this linearized equation of motion (LEOM) are expected to encompass all the modes of the open strings with endpoints on the D25–brane represented by Φ_0 as well as all the states which are $\mathcal{Q}_0$–exact.

To find the solutions to (5.3) we follow [23], but we introduce some significant changes: the dressing and the midpoint regularization. The ansatz for a general solution of momentum p is as follows

$$|\hat{\phi}_e(\mathcal{P}, \mathbf{t}, p)\rangle = \mathcal{N}_e \mathcal{P}(a^\dagger) \exp[-\sum_{n\geq 1} t_n a_n^{\mu\dagger}\hat{p}_\mu]|\hat{\Xi}_e\rangle \otimes |\hat{\widetilde{\Xi}}\rangle e^{ipx} \equiv |\varphi_e(\mathcal{P}, \mathbf{t}, p)\rangle \otimes |\hat{\widetilde{\Xi}}\rangle \tag{5.4}$$

where $\mathbf{t} = \{t_n\}$, $\mathcal{P}(a^\dagger)$ is some polynomial of expressions of the type $\sum_n \zeta_n a_n^\dagger$, and

$$\hat{p}\,\mathrm{e}^{ipx} = p\,\mathrm{e}^{ipx}, \qquad bpz(\hat{p}) = -\hat{p}$$

We will often drop the labels $\mathbf{t}, \mathcal{P}$ and p when no ambiguities are possible. The factorized form of (5.4) allows us to split the linearized equation of motion into ghost and matter part

$$\mathcal{Q}|\widehat{\hat{\Xi}}\rangle + |\widehat{\hat{\Xi}}\rangle *_g |\widehat{\hat{\Xi}}\rangle = 0 \tag{5.5}$$

$$|\hat{\varphi}_e\rangle = |\hat{\Xi}\rangle *_m |\hat{\varphi}_e\rangle + |\hat{\varphi}_e\rangle *_m |\hat{\Xi}\rangle \tag{5.6}$$

The ghost part will remain the same throughout the paper, and from now on we simply forget it and concentrate on the matter part.

In the above equation $|\hat{\Xi}_e\rangle$ formally coincides with $|\hat{\Xi}_\epsilon\rangle$, with ϵ replaced by e. The reason for this seemingly bizarre change of notation is because the parameter e plays a different role from ϵ. While ϵ is a deformation parameter and we are only interested in the limit $\epsilon \to 1$ (recall that for $\epsilon \neq 0, 1$ $\hat{\Xi}_\epsilon$ is not a solution to (4.6)), we will find that the linearized equation of motion can be solved for any value of e. The reason of this lies in a result we found in the previous chapter,

$$|\hat{\Xi}_\epsilon\rangle * |\hat{\Xi}_e\rangle = |\hat{\Xi}_{\epsilon\star e}\rangle \tag{5.7}$$

$$\hat{\mathcal{N}_\epsilon} = \mathcal{N}\left(\frac{1+(1-\epsilon)\kappa}{\kappa+1}\right)^D, \qquad \hat{\mathcal{N}_e} = \mathcal{N}\left(\frac{1+(1-e)\kappa}{\kappa+1}\right)^D$$

and

$$\epsilon \star e = \frac{\epsilon e}{1+(1-\epsilon)(1-e)\kappa} \tag{5.8}$$

The $\star$-multiplication is isomorphic to ordinary multiplication between real numbers: using the reparametrization

$$f_\epsilon = \frac{1+(1-\epsilon)\kappa}{\epsilon} = 1 + (\kappa+1)\frac{1-\epsilon}{\epsilon} \tag{5.9}$$

it is easy to check that $f_{\epsilon\star e} = f_\epsilon f_e$.

It is evident from (4.52) that

$$|\hat{\Xi}_e\rangle * |\hat{\Xi}\rangle = |\hat{\Xi}\rangle * |\hat{\Xi}_e\rangle = |\hat{\Xi}_e\rangle \tag{5.10}$$

for any value of the parameter e. This basic equality will allow us to construct solutions to the LEOM that contain the free parameter e. We anticipate that eventually, in order to guarantee finiteness of the three-tachyons coupling, e will have to be set to 1.

Let us see all this in more detail, i.e. let us find the general conditions for solving the LEOM (5.3). To this end we introduce the general state

$$|\hat{\varphi}_{e,\beta}\rangle = \mathcal{N}_e \exp\Big[-\sum_{n\geq 1} t_n a_n^{\mu\dagger}\hat{p}_\mu - \sum_{n\geq 1}\beta_n^\mu a_n^{\nu\dagger}\eta_{\mu\nu}\Big]|\hat{\Xi}_e\rangle\, \mathrm{e}^{ipx} \tag{5.11}$$

where, with respect to (5.4), we have inserted the parameters β_n^μ. By differentiating with respect to it the appropriate number of times and setting afterwards $\beta_n^\mu = 0$, we will be able to generate any polynomial in $a_n^\dagger$ and therefore reproduce any state of the form (5.4).

Now we need

$$\begin{aligned}{}_1\langle\hat{\Xi}_\epsilon|_2\langle\hat{\varphi}_{e,\beta}|V_3\rangle &= \frac{\hat{\mathcal{N}}_\epsilon\hat{\mathcal{N}}_e}{(\det\hat{\mathcal{K}}_{\epsilon e})^{\frac{D}{2}}}\exp\left[-\chi^T\hat{\mathcal{K}}_{\epsilon e}^{-1}\lambda - \frac{1}{2}\chi^T\hat{\mathcal{K}}_{\epsilon e}^{-1}\chi - \frac{1}{2}\lambda^T\mathcal{V}\hat{\mathcal{K}}_{\epsilon e}^{-1}\lambda\right]\\ &\cdot \exp\left[-\frac{1}{2}\sum_{n,m\geq 1} a_n^{(3)\dagger}V_{n,m}^{33}a_m^{(3)\dagger} - a_n^{(3)\dagger}(\mathbf{v}_{0n}-\mathbf{v}_{+n})p\right]|0\rangle_3\mathrm{e}^{-pV_{00}p}\,\mathrm{e}^{ipx}\end{aligned} \tag{5.12}$$

where we introduced

$$\hat{\mathcal{K}}_{\epsilon e} = 1 - \hat{\mathcal{S}}_{\epsilon e}\mathcal{V}, \qquad \hat{\mathcal{S}}_{\epsilon e} = \begin{pmatrix}\hat{S}_\epsilon & 0\\ 0 & \hat{S}_e\end{pmatrix}$$

together with

$$\chi = \begin{pmatrix} V^{21}a^{(3)\dagger} + p(\mathbf{v}_+ - \mathbf{v}_-)\\ V^{12}a^{(3)\dagger} + p(\mathbf{v}_- - \mathbf{v}_0)\end{pmatrix}, \qquad \lambda = C\begin{pmatrix}0\\ \beta - p\mathbf{t}\end{pmatrix} \tag{5.13}$$

In all these formulas we have introduced infinite vectors β^μ, $\mathbf{t}$, $\mathbf{v}_0$, $\mathbf{v}_+$, $\mathbf{v}_-$ with components

$$\beta_n^\mu, \qquad t_n, \qquad \mathbf{v}_{0n} = V_{0n}^{11} = V_{0n}^{22}, \qquad \mathbf{v}_{+n} = V_{0n}^{12}, \qquad \mathbf{v}_{-n} = V_{0n}^{21}, \tag{5.14}$$

respectively. We are interested in the above formula in the limit $\epsilon \to 1$, while keeping e fixed.

Let us recall from Appendix C that

$$\begin{aligned}\hat{\mathcal{N}}_\epsilon &= [\mathrm{Det}(1-\Sigma\mathcal{V})]^{D/2}\left(\frac{f_\epsilon}{\kappa + f_\epsilon}\right)^D\\ \mathrm{Det}(1-\hat{\Sigma}_{\epsilon e}\mathcal{V}) &= \left(\frac{\kappa + f_\epsilon f_e}{(\kappa+f_\epsilon)(\kappa+f_e)}\right)^2\mathrm{Det}(1-\Sigma\mathcal{V})\end{aligned}$$

from which we get the important relation

$$\lim_{\epsilon\to 1}\frac{\hat{\mathcal{N}}_\epsilon}{(\sqrt{\det\hat{\mathcal{K}}_{\epsilon e}})^D} = \lim_{f_\epsilon\to 1}\left(\frac{f_\epsilon(\kappa+f_e)}{\kappa + f_\epsilon f_e}\right)^D = 1 \tag{5.15}$$

To start with, let us consider the simplest example, i.e. $\beta = 0$, which means $\mathcal{P}(a^\dagger) = 1$ in (5.4) and define the candidate for the tachyon wavefunction. We will denote $\hat{\varphi}_e(1,\mathbf{t},p)$ by $\hat{\varphi}_e(\mathbf{t},p)$ or simply by $\hat{\varphi}_e$. We find that (5.12) takes the following form

$${}_1\langle\hat{\Xi}|_2\langle\hat{\varphi}_e|V_3\rangle = \lim_{\epsilon\to 1}{}_1\langle\hat{\Xi}_\epsilon|_2\langle\hat{\varphi}_e|V_3\rangle = \exp\left[-\mathbf{t}\,a^\dagger p - \frac{1}{2}G_1\,p^2\right]|\hat{\Xi}_e\rangle\,\mathrm{e}^{ipx} \tag{5.16}$$

where $\mathbf{t}$ is a solution to

$$\mathbf{t} = \mathbf{v}_0 - \mathbf{v}_+ + (V^{12}, V^{21})\,\hat{\mathcal{K}}_{1e}^{-1}\,\hat{\mathcal{S}}_{1e}\begin{pmatrix}\mathbf{v}_+ - \mathbf{v}_- \\ \mathbf{v}_- - \mathbf{v}_0\end{pmatrix} + (V^{12}, V^{21})\,\hat{\mathcal{K}}_{1e}^{-1}\,C\begin{pmatrix}0 \\ \mathbf{t}\end{pmatrix} \tag{5.17}$$

and

$$\begin{aligned} G_1 = 2V_{00} &+ (\mathbf{v}_+ - \mathbf{v}_-, \mathbf{v}_- - \mathbf{v}_0)\,\hat{\mathcal{K}}_{1e}^{-1}\,\hat{\mathcal{S}}_{1e}\begin{pmatrix}\mathbf{v}_+ - \mathbf{v}_- \\ \mathbf{v}_- - \mathbf{v}_0\end{pmatrix} \\ &+ 2(\mathbf{v}_+ - \mathbf{v}_-, \mathbf{v}_- - \mathbf{v}_0)\,\hat{\mathcal{K}}_{1e}^{-1}\,C\begin{pmatrix}0 \\ \mathbf{t}\end{pmatrix} + (0, \mathbf{t})C\,\mathcal{V}\,\hat{\mathcal{K}}_{1e}^{-1}\,C\begin{pmatrix}0 \\ \mathbf{t}\end{pmatrix} \end{aligned} \tag{5.18}$$

where $\hat{\mathcal{K}}_{1e}$ and $\hat{\mathcal{S}}_{1e}$ equal $\hat{\mathcal{K}}_{\epsilon e}$ and $\hat{\mathcal{S}}_{\epsilon e}$ when $\epsilon = 1$, respectively.

If we repeat the same derivation for the other star product, we find

$$_1\langle\hat{\varphi}_e|_2\langle\hat{\Xi}|V_3\rangle = \lim_{\epsilon\to 1}\,_1\langle\hat{\varphi}_e|_2\langle\hat{\Xi}_\epsilon|V_3\rangle = \exp\left[-\mathbf{t}\,a^\dagger p - \frac{1}{2}G_2\,p^2\right]|\hat{\Xi}_e\rangle\,\mathrm{e}^{ipx} \tag{5.19}$$

where, this time, $\mathbf{t}$ is a solution to

$$\mathbf{t} = \mathbf{v}_0 - \mathbf{v}_- - (V^{12}, V^{21})\,\hat{\mathcal{K}}_{e1}^{-1}\,\hat{\mathcal{S}}_{e1}\begin{pmatrix}\mathbf{v}_0 - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_-\end{pmatrix} + (V^{12}, V^{21})\,\hat{\mathcal{K}}_{e1}^{-1}\,C\begin{pmatrix}\mathbf{t} \\ 0\end{pmatrix} \tag{5.20}$$

and

$$\begin{aligned} G_2 = 2V_{00} &+ (\mathbf{v}_0 - \mathbf{v}_+, \mathbf{v}_+ - \mathbf{v}_-)\,\hat{\mathcal{K}}_{e1}^{-1}\,\hat{\mathcal{S}}_{e1}\begin{pmatrix}\mathbf{v}_0 - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_-\end{pmatrix} \\ &- 2(\mathbf{v}_0 - \mathbf{v}_+, \mathbf{v}_+ - \mathbf{v}_-)\,\hat{\mathcal{K}}_{e1}^{-1}\,C\begin{pmatrix}\mathbf{t} \\ 0\end{pmatrix} + (\mathbf{t}, 0)C\,\mathcal{V}\,\hat{\mathcal{K}}_{e1}^{-1}\,C\begin{pmatrix}\mathbf{t} \\ 0\end{pmatrix} \end{aligned} \tag{5.21}$$

where $\hat{\mathcal{K}}_{e1}$ and $\hat{\mathcal{S}}_{e1}$ equal $\hat{\mathcal{K}}_{e\epsilon}$ and $\hat{\mathcal{S}}_{e\epsilon}$ when $\epsilon = 1$, respectively.

The two couples of expressions (5.17,5.20) and (5.18,5.21) are formally different. Of course they must give rise to the same result. If we require twist invariance for $\mathbf{t}$, i.e. $C\mathbf{t} = \mathbf{t}$, it is easy to see that the two couples of equations collapse to a single one. However, for reasons that will become clear later on, we will not require twist invariance for $\mathbf{t}$ (see section 5.3 for more comments on this point). This is why we wrote the two couples of equations explicitly. In general, therefore, $\mathbf{t} = \mathbf{t}_+ + \mathbf{t}_-$. Hermiticity of the string field requires that $C\mathbf{t} = \mathbf{t}^*$, i.e. $\mathbf{t}_+^* = \mathbf{t}_+$ and $\mathbf{t}_-^* = -\mathbf{t}_-$.

We remark now that, if the above equations have a nontrivial solution for $\mathbf{t}$ and

$$e^{-\frac{1}{2}Gp^2} = \frac{1}{2}, \tag{5.22}$$

where $G = G_1 = G_2$, then $|\hat{\varphi}_e\rangle$ is a solution to the LEOM (5.6).

We also notice, for future use, that for a state of the general form (5.4) to satisfy the LEOM, the equation for $\mathbf{t}$ and G remain the same. The presence of a polynomial $\mathcal{P}(a^\dagger)$ does not affect the exponents, but only implies new conditions for the parameters in $\mathcal{P}(a^\dagger)$ (see below).

5.2 Solution for $\mathbf{t}$ and G

In this section we study the solutions to eqs.(5.17,5.20) and evaluate G. Since, due to the structure of these equations, *a priori* one cannot exclude the possibility of a singularity in $1-\epsilon$, we insert ϵ at the right places and take the limit $\epsilon \to 1$ on the solution.

5.2.1 The solutions for $\mathbf{t}$

Let us see first the relation between these two equations. We write $\mathbf{t} = \mathbf{t}_+ + \mathbf{t}_-$, where $C\mathbf{t}_\pm = \pm\mathbf{t}_\pm$ and apply C to (5.17). Keeping track of the ϵ dependence, we obtain

$$\mathbf{t}_+ - \mathbf{t}_- = \mathbf{v}_0 - \mathbf{v}_- + (X_+, X_-)\,\hat{\mathcal{K}}_{\epsilon e}^{-1}\,\hat{\mathcal{T}}_{\epsilon e}\begin{pmatrix} \mathbf{v}_- - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_0 \end{pmatrix} + (X_+, X_-)\,\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_+ - \mathbf{t}_- \end{pmatrix} \tag{5.23}$$

Doing the same with (5.20) we get

$$\mathbf{t}_+ - \mathbf{t}_- = \mathbf{v}_0 - \mathbf{v}_+ - (X_+, X_-)\,\hat{\mathcal{K}}_{e\epsilon}^{-1}\,\hat{\mathcal{T}}_{e\epsilon}\begin{pmatrix} \mathbf{v}_0 - \mathbf{v}_- \\ \mathbf{v}_- - \mathbf{v}_+ \end{pmatrix} + (X_+, X_-)\,\hat{\mathcal{K}}_{e\epsilon}^{-1}\begin{pmatrix} \mathbf{t}_+ - \mathbf{t}_- \\ 0 \end{pmatrix} \tag{5.24}$$

Next we introduce the operator σC, where $\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We have

$$(\sigma C)^2 = 1, \qquad (\sigma C)\,\hat{\mathcal{K}}_{e\epsilon}^{-1}\,(\sigma C) = \hat{\mathcal{K}}_{\epsilon e}^{-1}, \qquad (\sigma C)\,\hat{\mathcal{T}}_{e\epsilon}\,(\sigma C) = \hat{\mathcal{T}}_{\epsilon e} \tag{5.25}$$

Therefore, by suitably inserting $(\sigma C)^2$ in (5.24), applying the above transformations and applying C to the resulting equation we find

$$\mathbf{t}_+ + \mathbf{t}_- = \mathbf{v}_0 - \mathbf{v}_- + (X_+, X_-)\,\hat{\mathcal{K}}_{\epsilon e}^{-1}\,\hat{\mathcal{T}}_{\epsilon e}\begin{pmatrix} \mathbf{v}_- - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_0 \end{pmatrix} + (X_+, X_-)\,\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_+ + \mathbf{t}_- \end{pmatrix} \tag{5.26}$$

Taking the sum and the difference of (5.23) and (5.26) we find separate equations for $\mathbf{t}_+$ and $\mathbf{t}_-$:

$$\mathbf{t}_+ = \mathbf{v}_0 - \mathbf{v}_- + (X_+, X_-)\,\hat{\mathcal{K}}_{\epsilon e}^{-1}\,\hat{\mathcal{T}}_{\epsilon e}\begin{pmatrix} \mathbf{v}_- - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_0 \end{pmatrix} + (X_+, X_-)\,\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_+ \end{pmatrix} \tag{5.27}$$

$$\mathbf{t}_- = (X_+, X_-)\,\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix} 0 \\ \mathbf{t}_- \end{pmatrix} \tag{5.28}$$

Now we have to solve these two equations. The rather lenghty calculations are left for Appendix D. From the results therein one can see that, for $\epsilon = 1$ and setting $\mathbf{t}_+ = \mathbf{t}_0 + \mathbf{t}_\alpha$, the first equation reduces to

$$\mathbf{t}_0 = 3\frac{T^2 - T + 1}{T + 1}\mathbf{v}_0 \tag{5.29}$$

$$\left[1 - \frac{1}{\kappa + f_e}(|\xi\rangle + |C\xi\rangle)\langle\xi|\frac{f_e + T}{1 - T^2}\right]|\mathbf{t}_\alpha\rangle = 0 \tag{5.30}$$

where $\mathbf{t}_0$ is the result obtained in [23] (multiplied by $\sqrt{2}$). It is easy to see that (5.30) has the general solution

$$\mathbf{t}_\alpha = \alpha\,\langle\xi|\frac{1}{T+1}|\mathbf{t}_0\rangle\,(1 + C)\xi \tag{5.31}$$

for any number α. The factor $\langle\xi|\frac{1}{T+1}|\mathbf{t}_0\rangle$ has been introduced for later convenience.

As for eq.(5.28) for $\epsilon = 1$ it has a nontrivial solution

$$\mathbf{t}_- = \beta(1-C)\xi \tag{5.32}$$

with arbitrary β. This solution turns out to have an important role (see below). In conclusion we can say that at $\epsilon = 1$ the solution for $\mathbf{t}$ can be written as

$$\mathbf{t} = \mathbf{t}_0 + \alpha\,\langle\xi|\frac{1}{T+1}|\mathbf{t}_0\rangle\,(1+C)\xi + \beta(1-C)\xi \tag{5.33}$$

for arbitrary constants α and β.

5.2.2 Calculation of G.

Once again, in order to compute G, we reintroduce the deformation parameter ϵ as in the previous section (see Appendix D). We rewrite eqs.(5.18,5.21) as follows

$$\begin{aligned} G_1 = {} & 2V_{00} + (\mathbf{v}_+ - \mathbf{v}_-, \mathbf{v}_- - \mathbf{v}_0)\,\hat{\mathcal{K}}_{\epsilon\epsilon}^{-1}\,\hat{\mathcal{T}}_{\epsilon\epsilon}\begin{pmatrix}\mathbf{v}_- - \mathbf{v}_+\\ \mathbf{v}_+ - \mathbf{v}_0\end{pmatrix} \\ & + 2(\mathbf{v}_+ - \mathbf{v}_-, \mathbf{v}_- - \mathbf{v}_0)\,\hat{\mathcal{K}}_{\epsilon\epsilon}^{-1}\begin{pmatrix}0\\ \mathbf{t}_+ - \mathbf{t}_-\end{pmatrix} + (0, \mathbf{t}_+ + \mathbf{t}_-)\,\mathcal{M}\,\hat{\mathcal{K}}_{\epsilon\epsilon}^{-1}\begin{pmatrix}0\\ \mathbf{t}_+ - \mathbf{t}_-\end{pmatrix}\end{aligned} \tag{5.34}$$

and

$$\begin{aligned} G_2 = {} & 2V_{00} + (\mathbf{v}_0 - \mathbf{v}_+, \mathbf{v}_+ - \mathbf{v}_-)\,\hat{\mathcal{K}}_{\epsilon\epsilon}^{-1}\,\hat{\mathcal{T}}_{\epsilon\epsilon}\begin{pmatrix}\mathbf{v}_0 - \mathbf{v}_-\\ \mathbf{v}_- - \mathbf{v}_+\end{pmatrix} \\ & - 2(\mathbf{v}_0 - \mathbf{v}_+, \mathbf{v}_+ - \mathbf{v}_-)\,\hat{\mathcal{K}}_{\epsilon\epsilon}^{-1}\begin{pmatrix}\mathbf{t}_+ - \mathbf{t}_-\\ 0\end{pmatrix} + (\mathbf{t}_+ + \mathbf{t}_-, 0)\,\mathcal{M}\,\hat{\mathcal{K}}_{\epsilon\epsilon}^{-1}\begin{pmatrix}\mathbf{t}_+ - \mathbf{t}_-\\ 0\end{pmatrix}\end{aligned} \tag{5.35}$$

Using (5.33) we obtain

$$G_1 = G_0 - 2\,(f_\epsilon - 1)\frac{\kappa + f_e}{\kappa + f_\epsilon f_e}\left[\alpha(1-\kappa\alpha)\left(\langle\mathbf{t}_0|\frac{1}{1+T}|\xi\rangle\right)^2 + \beta\left(\kappa\beta + \langle\mathbf{t}_0|\frac{1}{1+T}|\xi\rangle\right)\right] \tag{5.36}$$

and

$$G_2 = G_0 - 2\,(f_\epsilon - 1)\frac{\kappa + f_e}{\kappa + f_\epsilon f_e}\left[\alpha(1-\kappa\alpha)\left(\langle\mathbf{t}_0|\frac{1}{1+T}|\xi\rangle\right)^2 + \beta\left(\kappa\beta - \langle\mathbf{t}_0|\frac{1}{1+T}|\xi\rangle\right)\right] \tag{5.37}$$

Therefore, for $\epsilon = 1$ we obtain $G_1 = G_2 = G_0$. Naive manipulations of the relevant formulas lead to the result $G_0 = 0$. However G_0 contains two divergent terms, which need to be regularized. As shown by Hata et al. [23, 61, 62], using level truncation one obtains[1] $G_0 = 2\ln 2$.

[1] Our definitions for $\mathbf{t}$ and G differ from those in [23] by factors of $\sqrt{2}$ and 2, respectively, see Appendix A.

5.3 The tachyon and vector excitations

After a long preparation we are now ready to start the analysis of the fluctuations around the dressed sliver.

5.3.1 The tachyon excitation

From the results of the previous section it follows that string fields of the form

$$|\hat{\varphi}_e(\mathbf{t}, p)\rangle = \mathcal{N}_e \exp\Big(- \sum_{n\geq 1} t_n a_n^{\mu\dagger} \hat{p}_\mu \Big) |\hat{\Xi}_e\rangle \, \mathrm{e}^{ipx} \tag{5.38}$$

with $\mathbf{t}$ as in (5.33), satisfy the LEOM when the momentum fulfills the mass-shell condition $m^2 = -p^2 = -1$. This solution depends on three arbitrary parameters e, α and β. Eventually we shall see that in fact we have to set $e = 1$. As we will see, the other two parameters never enter the evaluation of physical quantities. There is one more question. We expect the tachyon to be represented by a twist-even state, and we already noticed that (5.38) does not have definite twist parity. We will see later that the twist odd part of the tachyon state does not in fact contribute to observables such as the 3 tachyon coupling, on the other this twist violation will be crucial in obtaining the transversality condition for the U(1) gauge field.

5.3.2 The vector excitation

Fluctuations other than the tachyon can be obtained by considering nontrivial polynomials in eq.(5.4). The polynomial will consist of sum of monomials of the type

$$d^{\mu_1 \dots \mu_p} \langle \zeta_1 a^\dagger_{\mu_1} \rangle \, \dots \, \langle \zeta_p a^\dagger_{\mu_p} \rangle \tag{5.39}$$

where $\langle \zeta_i a^{\mu_i \dagger} \rangle = \sum_{n>0} \zeta_{in} a_n^{\mu_i \dagger}$. As it turns out the ϵ–dependence is trivial as far as higher fluctuations are concerned, therefore we drop it throughout.

Let us find the level one state, corresponding to the massless vector. We start with the following ansatz for the matter part

$$|\hat{\varphi}_{e,v}(d^\mu, \mathbf{t}, p)\rangle = \mathcal{N}_v \mathcal{N}_e d^\mu \langle (1-C)\zeta a^\dagger_\mu \rangle \, e^{-\sum_{n\geq 1} t_n a_n^{\mu\dagger} \hat{p}_\mu} |\hat{\Xi}_e\rangle \mathrm{e}^{ipx} = \mathcal{N}_v d^\mu \langle (1-C)\zeta a^\dagger_\mu \rangle |\hat{\varphi}_e(\mathbf{t}, p)\rangle \tag{5.40}$$

with $\rho_2 \zeta = \zeta$ and $\rho_1 \zeta = 0$.

Using the results of Appendix D we obtain

$$\begin{aligned} |\hat{\varphi}_{e,v}\rangle * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * |\hat{\varphi}_{e,v}\rangle = e^{-\frac{1}{2} G\, p^2} \Big[d^\mu \langle (1-C)\zeta a^\dagger_\mu \rangle + \\ + \frac{1}{\kappa + f_e} \langle \xi | \frac{f_e + T}{1 - T^2} | \zeta \rangle d^\mu \langle (1-C)\xi a^\dagger_\mu \rangle + 2\beta \, (p \cdot d) \, \langle \xi | \frac{\kappa - T}{1 - T^2} | \zeta \rangle \Big] \mathcal{N}_v \, |\hat{\varphi}_e(\mathbf{t}, p)\rangle \end{aligned} \tag{5.41}$$

From this result we see that in order to satisfy the LEOM we have to assume that $p^2 = 0$ and to impose the transversality condition

$$p \cdot d = 0 \tag{5.42}$$

Therefore we recover the massless vector state with the correct transversality condition. This result is independent of the value of e. In order to satisfy the LEOM we also have to impose

$$\langle \xi | \frac{f_e + T}{1 - T^2} | \zeta \rangle = 0 \tag{5.43}$$

This is to be understood as a condition on the vector ζ and as such it is easy to comply with it. For reasons that will become clear later, eventually we will set $e = 1$. In this case (5.43) becomes simply

$$\langle \xi | \frac{1}{1 - T} | \zeta \rangle = 0$$

which is the condition of orthogonality to the extra eigenvector(s) of the dressed sliver (4.58). To conclude we remark that dressing is essential in order to obtain the transversality condition.

5.4 Probing the $k \sim 0$ region

Level truncation is a natural regularization in the SFT context. It permits many numerical computations, but it is very unwieldy if one wants to derive analytical results, the lack of analytical control being related to the impossibility of using the analytical machinery of the continuous basis. This is true in particular for the region around $k = 0$, i.e. the string midpoint region, which turns out to be crucial for higher level excitations. In this section we therefore introduce an analytic surrogate of level truncation, at least as far as the $k \sim 0$ region is concerned. It consists of a regulator which mimics the level truncation by regulating the singularities arising when the $k \sim 0$ region is probed but has the good feature of being defined on the continuous basis (hence permitting analytical control).

To this end the crucial issue is the eigenvalues distribution at $k \sim 0$. As proved in [28] this distribution is divergent, but can be regularized in large–L level truncation

$$\rho(k) = \frac{\ln L}{2\pi} + \rho_{fin}(k) \tag{5.44}$$

the quantity $\rho_{fin}(k)$ is responsible for finite contributions which are relevant for large k, see [50], but it will play no role in the sequel. The eigenvectors of the k–basis have infinite norm due to the continuous orthonormality condition

$$\langle k | k' \rangle = \delta(k - k') \tag{5.45}$$

Large–L level regularization suggests that their norm is given by[2]

$$\langle k|k\rangle = \delta(0) = \frac{\ln L}{2\pi} \tag{5.46}$$

Consider now the following half (right) string vector in the k–basis

$$|\eta\rangle = \frac{1}{\eta}\int_{\frac{\eta}{2}}^{\frac{3\eta}{2}} dk|k\rangle, \qquad \eta > 0 \tag{5.47}$$

The norm of this vector is easily computed to be

$$\langle \eta|\eta\rangle = \frac{1}{\eta} \tag{5.48}$$

From this we define a twist–even and a twist–odd vector as follows

$$\begin{aligned} |\eta_+\rangle &= \frac{1}{\sqrt{2}}\left(|\eta\rangle + C|\eta\rangle\right) \\ |\eta_-\rangle &= \frac{1}{\sqrt{2}}\left(|\eta\rangle - C|\eta\rangle\right) \end{aligned} \tag{5.49}$$

Their norm is given by

$$\langle \eta_-|\eta_-\rangle = \langle \eta_+|\eta_+\rangle = \frac{1}{\eta} \tag{5.50}$$

These two vectors are the basis of our regularization. In the limit $\eta \to 0^+$ they collapse to the midpoint $k = 0$, and keeping track of the powers of η will allow us to give an unambiguous meaning to the objects we are interested in.

Our first aim is to show that this procedure is inspired by and very close to the level truncation. To this end let us expand these two vectors in the oscillator basis $|n\rangle$. Using

$$\langle n|k\rangle = \sqrt{\frac{nk}{2\sinh\frac{\pi k}{2}}}\oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}}\frac{1}{k}\left(1 - \exp(-k\tan^{-1} z)\right)$$

a term by term integration yields

$$\begin{aligned} \langle n|\eta_-\rangle &= \sqrt{\frac{2}{\pi}}\left(1, 0, -\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{5}}, 0, ...\right) + O(\eta^2) \\ \langle n|\eta_+\rangle &= -\frac{\eta}{\sqrt{2\pi}}\left(0, \sqrt{2}, 0, -\frac{4}{3}, 0, \frac{23}{15}\sqrt{\frac{2}{3}}, 0, ...\right) + O(\eta^2) \end{aligned} \tag{5.51}$$

The first vector is therefore the usual $|k = 0\rangle$ twist–odd vector, while every component of the second vanishes in the limit $\eta \to 0$. The latter is $(-\eta\sqrt{2/\pi})$ times the $K^2 = 0$ twist–even vector Rastelli, Sen and Zwiebach found in [28], that is

$$|0_-\rangle = \lim_{\eta\to 0^+}|\eta_-\rangle = \sqrt{\frac{2}{\pi}}\,|v^-_{RSZ}\rangle \tag{5.52}$$

$$|0_+\rangle = \lim_{\eta\to 0^+}|\eta_+\rangle = -\eta\sqrt{\frac{2}{\pi}}\,|v^+_{RSZ}\rangle \tag{5.53}$$

[2] Again finite corrections are neglected, as they are not important for our purposes.

It is important to note that although the twist–even vector $|0_+\rangle$ is vanishing, due to (5.50), it has the same infinite norm as $|0_-\rangle$. Like all the vectors which form the continuous basis, this vector does not belong either to the Fock space, but, unlike all other $|k\rangle$'s, it has vanishing overlap with all oscillators

$$\langle n|0_+\rangle = \lim_{\eta\to 0}\langle n|\eta_+\rangle = 0 \tag{5.54}$$

Nevertheless, as we will see in the sequel, it is crucial for the consistency of the $*$–algebra and, moreover, for accommodating the complete open string D–brane spectrum in the VSFT approach.

At this stage it should be clear that the η parameter plays the role of an effective large L truncation of the continuous basis, and that $|\eta_-\rangle$ represents the eigenvector relative to the smallest eigenvalue of T at level $L(\eta)$, which is always twist–odd. From [62] we expect the first eigenvector to be located at $k = \frac{\pi}{\log L}$. This suggests that one should make the identification

$$\eta = \frac{\pi}{\log L} \tag{5.55}$$

We can verify this assertion by checking that

$$\langle 0_-|0_-\rangle = \langle 0_+|0_+\rangle$$

Using (5.52), this gives

$$\eta = \sqrt{\frac{\langle v^-_{RSZ}|v^-_{RSZ}\rangle}{\langle v^+_{RSZ}|v^+_{RSZ}\rangle}} \tag{5.56}$$

Computing the difference between the RHS of (5.55) and the RHS of (5.56) in level truncation we find that it becomes smaller and smaller as $L\to\infty$. For example at $L = 1000$ we have $\frac{\pi}{\log L}\sim 0.45479$ (not very near 0!) and such a difference is -0.03082, while at $L = 10000$ we have 0.34109 and -0.01040, respectively, which is a 3% agreement. Proceeding further with the level it is easy to verify that the agreement improves[3].

We have therefore succeeded in relating our regularization parameter η to the cut-off L. With some abuse of language we will call the previous empirical set of rules η–regularization. Now we are going to show that some ambiguities that used to plague the string midpoint analysis, within this regularization scheme are naturally resolved. We are interested, in particular, in the action of the half string projectors $\rho_{1,2}$ on the midpoint modes $|0_\pm\rangle$. By using the η–regularization (5.49) we simply get

$$\begin{aligned}\rho_1|0_\pm\rangle &= \frac{1}{2}|0_\pm\rangle + \frac{1}{2}|0_\mp\rangle\\ \rho_2|0_\pm\rangle &= \frac{1}{2}|0_\pm\rangle - \frac{1}{2}|0_\mp\rangle\\ (\rho_1-\rho_2)|0_\pm\rangle &= |0_\mp\rangle\end{aligned} \tag{5.57}$$

[3]This simple example should warn the reader on how level truncation is slow in probing the midpoint $k = 0$.

If we contract this result with any Fock space vector $\langle n|$, we recover the result of [24] that the ρ projectors have $\frac{1}{2}$ eigenvalue at $k = 0$. The latter assertion is however, by itself, not free from ambiguities and/or associativity inconsistencies if we do not want to give up the properties (A.28). For example, a naive manipulation leads to

$$0 = (\rho_1\rho_2)|0_-\rangle \neq \rho_1(\rho_2|0_-\rangle) = \frac{1}{4}|0_-\rangle \tag{5.58}$$

On the contrary, with our regularization it is very easy to check that

$$0 = (\rho_1\rho_2)|0_\pm\rangle = \rho_1(\rho_2|0_\pm\rangle) = 0 \tag{5.59}$$

which is definitely non–ambiguous. Other remarkable inconsistencies which would arise using the same kind of naive manipulations would be

$$\begin{aligned} \frac{1}{2}|0_-\rangle = (\rho_{1,2}\rho_{1,2})|0_-\rangle \neq \rho_{1,2}(\rho_{1,2}|0_-\rangle) = \frac{1}{4}|0_-\rangle \\ |0_-\rangle = (\rho_1-\rho_2)^2|0_-\rangle \neq (\rho_1-\rho_2)\left((\rho_1-\rho_2)|0_-\rangle\right) = 0 \end{aligned} \tag{5.60}$$

It is easy to check that, with our regularization, this anomaly disappears and all the properties (A.28) are preserved even at $k = 0$. The crucial move was to introduce an extra twist–even midpoint vector which vanishes in the Fock space, but has nevertheless infinite norm. We will see in the sequel how this vanishing vector is important for the construction of open string states on the dressed sliver. For the time being we only point out that the vector $|0_+\rangle$ cannot create string excitations when contracted with oscillators since, see (5.54),

$$\langle 0_+|a^\dagger\rangle|state\rangle = \lim_{\eta\to 0}\sum_n a_n^\dagger\,\langle n|\eta_+\rangle|state\rangle = 0 \tag{5.61}$$

vanishes. However we can excite Fock space states if, in η–regularization, we consider the vector

$$\lim_{\eta\to 0^+}\frac{1}{\eta}|\eta_+\rangle \sim |v^+_{RSZ}\rangle \tag{5.62}$$

From (5.51) it is clear that this vector has finite overlap with any Fock space vector. We will see that this vector plays a fundamental role in the construction of cohomologically non–trivial open string states. The vector $|0_+\rangle$ can also contribute to matrix elements involving vectors that are finite at the midpoint (hence out of the Fock space) like the "bare tachyon" $\langle t_0|$. For example the following relations hold in η–regularization

$$\langle t_0|0_+\rangle = \sqrt{2}\,t_0(0) + O(\eta) \tag{5.63}$$

$$\langle t_0|\frac{1}{1+T}|0_+\rangle = \ln 3\,\frac{2\sqrt{2}}{\pi}\,t_0(0)\frac{1}{\eta} + O(1) \tag{5.64}$$

In the sequel we will see that, using η–regularization, all the divergent brackets that appear in computing solutions to the LEOM can be explicitly evaluated in terms of some (regularization dependent) function of η. We will comment a posteriori on the regularization independence of our final and physical results.

5.5 Higher level solutions to LEOM

In the canonical quantization of string theory the tower of massive states is constructed by applying monomials of creation operators on the Fock vacuum. In order for the state to have a definite mass one selects all the monomials of the same level and takes a linear combination thereof, with tensorial coefficients which are generically referred to as polarizations. The latter are not completely free, but must satisfy some constraints, the Virasoro constraints. The construction of analogous states in VSFT proceeds differently. Although we will keep talking about level n solutions in order to relate our results with the familiar ones, the level is not the right issue here, because in VSFT we do not have any explicit realization of the L_0 Virasoro generator. The most general level n state we will consider will take the form

$$|\hat{\varphi}(\theta, n, \mathbf{t}, p)\rangle \equiv |\hat{\varphi}(\theta_1, ..., \theta_n, \mathbf{t}, p)\rangle = \sum_{i=1}^{n} \theta_i^{\mu_1 \ldots \mu_i} \langle a^\dagger_{\mu_1} \zeta_1^{(i)} \rangle \ldots \langle a^\dagger_{\mu_i} \zeta_i^{(i)} \rangle |\hat{\varphi}(\mathbf{t}, p)\rangle \tag{5.65}$$

in analogy with the canonical quantization construction, but without imposing any level restriction. As we shall see below, the request that the state (5.65) satisfy the LEOM will be sufficient to select a definite mass and impose the appropriate Virasoro constraints on the polarizations θ_i.

5.5.1 Level 2

The level 2 state in canonical quantization is

$$(h_\mu \alpha_2^{\mu\dagger} + \lambda_{\mu\nu} \alpha_1^{\mu\dagger} \alpha_1^{\nu\dagger})|0\rangle e^{ipx} \,. \tag{5.66}$$

The Virasoro constraints require that $p^2 = -1$ and

$$2\sqrt{2}\, h^\mu p_\mu + \lambda_\mu{}^\mu = 0\,, \qquad h_\mu + \sqrt{2}\, \lambda_{\mu\nu} p^\nu = 0 \tag{5.67}$$

In view of the forthcoming VSFT construction it is important to notice that there is a certain arbitrariness in these formulas. One can rewrite them for instance as follows

$$2\sqrt{2}\, g^\mu p_\mu + a\theta_\mu{}^\mu = 0\,, \qquad b g_\mu + \sqrt{2}\, \theta_{\mu\nu} p^\nu = 0 \tag{5.68}$$

with a and b arbitrary (non–vanishing) constants, and h, λ related to g, θ as follows

$$h_\mu = A g_\mu + B(p\cdot g) p_\mu, \qquad \lambda_{\mu\nu} = C\theta_{\mu\nu} + D(p_\mu p^\rho \theta_{\rho\nu} + p_\nu p^\rho \theta_{\mu\rho}) \tag{5.69}$$

Using the mass–shell condition it is easy to show that this simply requires

$$A = \frac{b}{2}\,\frac{3ab+2}{ab-1}\, D\,, \qquad B = b\, D\,, \qquad C = \frac{5}{2}\,\frac{ab}{ab-1}\, D$$

According to the level n ansatz (5.65) the candidate to represent a level 2 state is

$$|\hat{\varphi}(\theta, 2, \mathbf{t}, p)\rangle \equiv |\hat{\varphi}(\theta_1, \theta_2, \mathbf{t}, p)\rangle = \theta_1^{\mu_1} \langle a_{\mu_1}^\dagger \zeta_1^{(1)}\rangle |\hat{\varphi}(\mathbf{t}, p)\rangle + \theta_2^{\mu_1\mu_2} \langle a_{\mu_1}^\dagger \zeta_1^{(2)}\rangle \langle a_{\mu_2}^\dagger \zeta_2^{(2)}\rangle |\hat{\varphi}(\mathbf{t}, p)\rangle \tag{5.70}$$

This ansatz has to be made more precise by specifying the vectors $|\zeta_j^{(i)}\rangle$. For generic vectors we do not get any on–shell open string state. In fact, on the basis of our attempts, it seems that only if the vectors $|\zeta_j^{(i)}\rangle$ probe the string midpoint will (5.70) be a cohomologically non–trivial solution to the LEOM. Therefore we make the choice $|\zeta_j^{(i)}\rangle \sim |0_\pm\rangle$; the latter states were introduced in the previous section and were designed to resolve the singularity at $k = 0$. But we must be more precise: the factors in front of $\lim_{\eta\to 0^+} |\eta_\pm\rangle$ play also a fundamental role and we must specify them. In summary, our ansatz will be

$$|\hat{\varphi}(g, \theta, \mathbf{t}, p)\rangle = g^\mu \langle a_\mu^\dagger | s_+\rangle |\hat{\varphi}(\mathbf{t}, p)\rangle + \theta^{\mu\nu} \langle a_\mu^\dagger | \zeta_-\rangle \langle a_\nu^\dagger | \zeta_-\rangle |\hat{\varphi}(\mathbf{t}, p)\rangle \tag{5.71}$$

where $|s_+\rangle = \lim_{\eta\to 0^+} |\eta_+\rangle s(\eta)$, $|\zeta_-\rangle = \lim_{\eta\to 0^+} |\eta_-\rangle \zeta(\eta)$, and, near $\eta = 0$,

$$s(\eta) = \frac{s_{-1}}{\eta} + s_0 + s_1\eta + \ldots, \qquad \zeta(\eta) = \zeta_0 + \zeta_1\eta + \zeta_2\eta^2 + \ldots \tag{5.72}$$

As a consequence we have (see (5.52,5.53))

$$\langle a_\mu^\dagger | s_+\rangle = -\sqrt{\frac{2}{\pi}} \langle a_\mu^\dagger | v_{RSZ}^+\rangle \left(s_{-1} + s_0\eta + s_1\eta^2 + \ldots\right) \tag{5.73}$$

$$\langle a_\mu^\dagger | \zeta_-\rangle = \sqrt{\frac{2}{\pi}} \langle a_\mu^\dagger | v_{RSZ}^-\rangle \left(\zeta_0 + \zeta_1\eta + \zeta_2\eta^2 + \ldots\right) \tag{5.74}$$

These are well–defined expressions and it would seem that the terms proportional to η, η^2 play no role in the limit $\eta \to 0$. However this is not the case because the star product with the dressed sliver will take them back into the game. Only terms of order η^3 and higher will not play any role and can be disregarded.

It is time to pass to the explicit calculation of the LEOM. We have to find the conditions under which

$$|\hat{\varphi}(g, \theta, \mathbf{t}, p)\rangle * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * |\hat{\varphi}(g, \theta, \mathbf{t}, p)\rangle = |\hat{\varphi}(g, \theta, \mathbf{t}, p)\rangle \tag{5.75}$$

The star products in (5.75) yield cumbersome formulas. In order not to clog our exposition with them we defer a full treatment to Appendix D, and use a technical simplification: we assume that the function $\xi(k)$, which represents the dressing vector ξ in the k–basis and which is non-vanishing only for negative k, is actually non–vanishing only for $k < k_0 < 0$ where k_0 is some small but finite negative constant. The consequences of this simplification will be commented upon in section 6 of the present chapter. We can of course suppose that the regularization parameter $2\eta < |k_0|$. As a consequence all the quantities appearing in this computation which involve ξ can be neglected. On the other hand this restriction on the form of $\xi(k)$ does not imperil the properties we have requested for ξ in all the results we have so far obtained. With this understanding we obtain

$$\begin{aligned}
&\left(\theta^{\mu\nu}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle|\hat\varphi(\mathbf{t},p)\rangle\right)*|\hat\Xi\rangle+|\hat\Xi\rangle*\left(\theta^{\mu\nu}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle|\hat\varphi(\mathbf{t},p)\rangle\right) \quad (5.76)\\
&= e^{-\frac{1}{2}Gp^2}\left[\frac{1}{2}\theta^{\mu\nu}\,\langle a_\mu^\dagger|\zeta_-\rangle\,\langle a_\nu^\dagger|\zeta_-\rangle+2\,\theta_\mu{}^\mu\,\langle\zeta_-|\frac{T}{1-T^2}|\zeta_-\rangle\right.\\
&\qquad\left.+\,2\,\theta^{\mu\nu}\,\langle a_\mu^\dagger|\zeta_+\rangle\,p_\nu\,\mathcal{H}_+ + 2\theta^{\mu\nu}p_\mu p_\nu\mathcal{H}_+^2\right]|\hat\varphi(\mathbf{t},p)\rangle
\end{aligned}$$

where we have used $|\zeta_+\rangle=(\rho_1-\rho_2)|\zeta_-\rangle$, with (5.61), and

$$\begin{aligned}
&\left(g^\mu\langle a_\mu^\dagger|s_+\rangle|\hat\varphi(\mathbf{t},p)\rangle\right)*|\hat\Xi\rangle+|\hat\Xi\rangle*\left(g^\mu\langle a_\mu^\dagger|s_+\rangle|\hat\varphi(\mathbf{t},p)\rangle\right)\\
&\qquad= e^{-\frac{1}{2}Gp^2}\left[g^\mu\langle a_\mu^\dagger|s_+\rangle-p\cdot g\,\langle t_0|\frac{1}{1+T}|s_+\rangle\right]|\hat\varphi(\mathbf{t},p)\rangle
\end{aligned}\tag{5.77}$$

The quantity $\mathcal{H}_+$ (see appendix D) is a complicated expression of order η^{-1}, $\langle\zeta_-|\frac{T}{1-T^2}|\zeta_-\rangle$, as well as $\langle t_0|\frac{1}{1+T}|s_+\rangle$, is of order η^{-2}, while, as we have already seen, $\langle a_\mu^\dagger|\zeta_+\rangle$ is of order η.

Now, from the first term in the RHS of eq.(5.76) we see that the only way to satisfy the LEOM is to set $e^{-\frac{1}{2}Gp^2}=2$, i.e. $p^2=-1$, which reproduces the desired mass–shell condition. Next, in (5.77) we must split $g^\mu\langle a_\mu^\dagger|s_+\rangle$ (which is a finite term in η) in two halves. The first half reconstructs the first term in the RHS of (5.71), the second half must annihilate the linear term in $a^\dagger$ in the RHS of (5.76): this is the only way this unwanted term can be canceled. The latter operation on the other hand is only possible if

$$g^\mu\sim\theta^{\mu\nu}p_\nu \tag{5.78}$$

Finally the remaining unwanted terms in the above equations must cancel with one another order by order in η. Looking at the order –2 in η, one easily realizes that the only way to implement such cancelation is to require that

$$\theta_\mu{}^\mu\sim\theta^{\mu\nu}p_\mu p_\nu\sim p\cdot g \tag{5.79}$$

with nonvanishing proportionality constants.

Eqs.(5.79,5.78) are not enough to conclude that the level 2 Virasoro constraints (5.68) are satisfied. However the accurate analysis of Appendix D proves that this is the case. In Appendix D it is also shown that the LEOM (5.75) is exactly satisfied together with the Virasoro constraints (5.68), provided some (not very restrictive) relations among the constants $a, b, \zeta_0, \zeta_1, \zeta_2, s_{-1}, s_0, s_1$ are satisfied. From the analysis in Appendix D it is clear that the coefficients a and b are regularization dependent, but, in turn, a and b can be absorbed via the redefinitions (5.69).

5.5.2 Level 3

The level 3 state in canonical quantization is

$$(h_\mu \alpha_3^{\mu\dagger} + \lambda_{\mu\nu}\alpha_2^{\mu\dagger}\alpha_1^{\nu\dagger} + \chi_{\mu\nu\rho}\alpha_1^{\mu\dagger}\alpha_1^{\nu\dagger}\alpha_1^{\rho\dagger})|0\rangle e^{ipx}, \tag{5.80}$$

The Virasoro constraints require that $p^2 = -2$ and

$$3h^\mu p_\mu + \sqrt{2}\,\lambda_\mu{}^\mu = 0\,, \qquad 3h_\mu + \sqrt{2}\,\lambda_{\mu\nu}p^\nu = 0 \tag{5.81}$$

$$\sqrt{2}\,(2\,\lambda_{\nu\mu}p^\nu - \lambda_{\mu\nu}p^\nu) + 3\chi_{\mu\nu}{}^\nu = 0\,, \qquad \sqrt{2}\,\lambda_{(\mu\nu)} + 3\chi_{\mu\nu\rho}p^\rho = 0 \tag{5.82}$$

where $\lambda_{(\mu\nu)}$ is the symmetric part of $\lambda_{\mu\nu}$. It can be seen that the first equation is a consequence of the other three. It is however possible, as above, to redefine the polarizations as shown in Appendix D. In terms of the new ones $g_\mu, \omega_{\mu\nu}, \theta_{\mu\nu\rho}$ the Virasoro constraints become

$$3\,x\,g^\mu p_\mu + \sqrt{2}\,\omega_\mu{}^\mu = 0\,, \qquad 3\,g_\mu + \sqrt{2}\,y\,\omega_{\mu\nu}p^\nu = 0 \tag{5.83}$$

$$2\sqrt{2}\,v\,\omega_{\nu\mu}p^\nu - \sqrt{2}\,u\,\omega_{\mu\nu}p^\nu + 3\,\theta_{\mu\nu}{}^\nu = 0\,, \qquad \sqrt{2}\,\omega_{(\mu\nu)} + 3\,z\,\theta_{\mu\nu\rho}p^\rho = 0 \tag{5.84}$$

It is now easy to verify that the first condition is a consequence of the other three provided we set $x = \frac{z(2v-u)}{y}$. Therefore it need not be verified separately. The remaining constants y, u, v, z are arbitrary non–vanishing ones. From the general form (5.65), we select the following ansatz

$$\begin{aligned} &|\hat\varphi(g,\omega,\theta,\mathbf{t},p)\rangle \\ &\quad = \Big(g^\mu \langle a_\mu^\dagger|r_-\rangle + \omega^{\mu\nu}\langle a_\mu^\dagger|\zeta'_-\rangle\langle a_\nu^\dagger|\lambda_+\rangle + \theta^{\mu\nu\rho}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle\langle a_\rho^\dagger|\zeta_-\rangle\Big)\,|\hat\varphi(\mathbf{t},p)\rangle \end{aligned} \tag{5.85}$$

where $|r_-\rangle = \lim_{\eta\to 0^+}|\eta_-\rangle r(\eta)$, and the same definition is understood for $|\zeta'_-\rangle, |\zeta_-\rangle$, while $|\lambda_+\rangle = \lim_{\eta\to 0^+}|\eta_+\rangle\lambda(\eta)$. Near $\eta = 0$,

$$\lambda(\eta) = \frac{\lambda_{-1}}{\eta} + \lambda_0 + \lambda_1\eta + \ldots, \qquad \zeta(\eta) = \zeta_0 + \zeta_1\eta + \zeta_2\eta^2 + \ldots \tag{5.86}$$

$\zeta'(\eta)$ and $r(\eta)$ have an expansion similar to $\zeta(\eta)$. Consequently, for the brackets inside (5.85), expansions similar to (5.73) and (5.74) hold.

The formulas involved in the evaluation of the linearized EOM are too large to be written down here. We can avoid such complications by introducing the simplifying assumption of the previous subsection: we render the dressing vector contributions evanescent in the limit $\eta \to 0$ so that we can simply avoid writing them down. The resulting formulas are as follows:

$$\begin{aligned} &\Big(\theta^{\mu\nu\rho}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle\langle a_\rho^\dagger|\zeta_-\rangle|\hat\varphi(\mathbf{t},p)\rangle\Big) * |\hat\Xi\rangle + |\hat\Xi\rangle * \Big(\theta^{\mu\nu\rho}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle\langle a_\rho^\dagger|\zeta_-\rangle|\hat\varphi(\mathbf{t},p)\rangle\Big) \\ &= e^{-\frac{1}{2}Gp^2}\Big[3\,\theta_\mu{}^{\mu\rho}\,\langle\zeta_-|\frac{T}{1-T^2}|\zeta_-\rangle\langle a_\rho^\dagger|\zeta_-\rangle + \theta^{\mu\nu\rho}\,\Big(\frac{1}{4}\langle a_\mu^\dagger|\zeta_-\rangle\,\langle a_\nu^\dagger|\zeta_-\rangle\langle a_\rho^\dagger|\zeta_-\rangle \\ &\qquad +3\,\langle a_\mu^\dagger|\zeta_-\rangle p_\nu\,p_\rho\mathcal{H}_+^2 + 3\,\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_+\rangle p_\rho\mathcal{H}_+\Big)\Big]\,|\hat\varphi(\mathbf{t},p)\rangle \end{aligned} \tag{5.87}$$

$$
\begin{aligned}
&\left(\omega^{\mu\nu}\langle a^\dagger_\mu|\zeta'_-\rangle\langle a^\dagger_\nu|\lambda_+\rangle|\hat\varphi(\mathbf{t},p)\rangle\right) * |\hat\Xi\rangle + |\hat\Xi\rangle * \left(\omega^{\mu\nu}\langle a^\dagger_\mu|\zeta'_-\rangle\langle a^\dagger_\nu|\lambda_+\rangle|\hat\varphi(\mathbf{t},p)\rangle\right) \qquad (5.88)\\
&= e^{-\frac{1}{2}Gp^2}\omega^{\mu\nu}\left[\frac{1}{2}\,\langle a^\dagger_\mu|\zeta'_-\rangle\,\langle a^\dagger_\nu|\lambda_+\rangle + \frac{1}{2}\,\langle a^\dagger_\mu|\zeta'_+\rangle\,\langle a^\dagger_\nu|\lambda_-\rangle\right.\\
&\qquad\left. + \langle a^\dagger_\mu|\zeta'_-\rangle p_\nu\,\langle \mathbf{t}_0|\frac{T}{1-T^2}|\lambda_+\rangle + p_\mu\,\langle a^\dagger_\nu|\lambda_-\rangle\,\mathcal{H}_+\right]|\hat\varphi(\mathbf{t},p)\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(g^\mu\langle a^\dagger_\mu|r_-\rangle|\hat\varphi(\mathbf{t},p)\rangle\right) * |\hat\Xi\rangle + |\hat\Xi\rangle * \left(g^\mu\langle a^\dagger_\mu|r_-\rangle|\hat\varphi(\mathbf{t},p)\rangle\right) \qquad (5.89)\\
&\qquad = e^{-\frac{1}{2}Gp^2}\,g^\mu\langle a^\dagger_\mu|r_-\rangle|\hat\varphi(\mathbf{t},p)\rangle
\end{aligned}
$$

where $|\lambda_-\rangle = \lim_{\eta\to 0^+}|\eta_-\rangle\lambda(\eta)$, and $|\zeta_+\rangle = \lim_{\eta\to 0^+}|\eta_+\rangle\zeta(\eta)$, $\lambda(\eta)$ and $\zeta(\eta)$ being the same functions as above, (5.86).

Now, in order for the LEOM to be satisfied the sum of these three terms, (5.87, 5.88) and (5.89), must reproduce (5.85). From the second term in the RHS of (5.88) we see that we must have $e^{-\frac{1}{2}Gp^2} = 4$, i.e. $p^2 = -2$, the mass–shell condition for level 3 states. This implies that, the RHS of the second equation $\omega^{\mu\nu}\langle a^\dagger_\mu|\zeta'_-\rangle\,\langle a^\dagger_\nu|\lambda_+\rangle$ appears with a coefficient 2 in front, therefore half of this term will reproduce (5.85) and the other half must be canceled against the other terms. Similarly in the RHS of (5.89) the term $g^\mu\langle a^\dagger_\mu|r_-\rangle$ appears with a coefficient 4. So 1/4 of it will reproduce (5.85) and 3/4 will have to be canceled.

Next, as in the previous subsection, we count the degrees of divergence for $\eta\to 0$ of the various terms in the above three equations, which is -2 for the first and third terms of the RHS of (5.87) and 0 for the remaining ones; it is 0 for the first two terms in the RHS of (5.88) and –2 for the other two; finally it is zero for the term in the RHS of (5.89). Now what we have to do is collecting all the unwanted terms in the RHS and imposing that the sum of the coefficients in front of them vanish. From what we just said, we can deduce that we must have

$$
\begin{aligned}
&\omega^{\mu\nu} \sim \theta^{\mu\nu\rho}p_\rho\\
&\theta_\mu{}^{\mu\rho} \sim \theta^{\mu\nu\rho}p_\mu p_\nu \sim a\,\omega^{\mu\rho}p_\mu + b\,\omega^{\rho\mu}p_\mu\\
&\omega^{\mu\rho}p_\mu \sim g^\rho
\end{aligned}
\tag{5.90}
$$

for some constants a and b. These are very close to (5.83,5.84). However it must be proven that the arbitrary constants we have at our disposal (i.e. x, y, u, v, z and the coefficients of $\zeta(\eta), \lambda(\eta)$ and $r(\eta)$) are sufficient to satisfy all the conditions. This is an elementary algebraic problem. The straightforward calculations are carried out in Appendix D where it is shown that all the conditions are met. So we can conclude that

$$
|\hat\varphi(g,\omega,\theta,\mathbf{t},p)\rangle * |\hat\Xi\rangle + |\hat\Xi\rangle * |\hat\varphi(g,\omega,\theta,\mathbf{t},p)\rangle = |\hat\varphi(g,\omega,\theta,\mathbf{t},p)\rangle \tag{5.91}
$$

5.6 Cohomology

A solution to the LEOM is not automatically a solution fit to represent a physical string state. The reason for this is the huge gauge invariance which soaks all physical states in SFT. Any solution to the LEOM is in fact defined up to

$$\mathcal{Q}_0\Lambda \equiv \mathcal{Q}\Lambda + \Phi_0 * \Lambda - \Lambda * \Phi_0 \tag{5.92}$$

where Φ_0 is our reference classical solution (see section 3) and Λ is any string state of ghost number 0. Only string field solutions which cannot take the form of (5.92) are significant solutions and can represent physical states. Phrased another way, $\mathcal{Q}_0$ is nilpotent, therefore it defines a cohomology problem: only nontrivial cohomology classes are physically interesting. Unfortunately a systematic approach to this problem is missing (although some progress can be found in [82]), the more so for VSFT. Partial elaborations on the gauge freedom in VSFT can be found in [24, 83]. In this section we will not try a systematic approach to the cohomology problem. Nevertheless it turns out to be rather easy to figure out Λ 'counterterms' that 'almost trivialize' the solutions we have found in the previous section, but actually do not kill them at all. This makes us confident that what we have found in the previous sections singles out nontrivial cohomology classes.

To simplify the problem as much as possible we will exclude all the Λ's with a nontrivial ghost content. If Λ is a matter state tensored with the ghost identity, see [24, 83], then the gauge transformation (5.92) for a (pure matter) state ϕ can be written simply through Λ's matter part as follows:

$$\delta\phi = \hat{\Xi} *_m \Lambda - \Lambda *_m \hat{\Xi} \tag{5.93}$$

where $\hat{\Xi}$ is the dressed sliver. Our problem is now to find matter states Λ such that (5.93) gives some of the solutions we found in the previous sections. Let us try the following one (we set $e = 1$ and drop the label m in $*_m$ throughout this section)

$$|\Lambda(g,\zeta)\rangle = g^\mu \langle (1+C)\zeta\, a_\mu^\dagger\rangle |\hat{\varphi}_t(\mathbf{t},p)\rangle \tag{5.94}$$

where $|\hat{\varphi}_t(\mathbf{t},p)\rangle$ is the tachyon wavefunction. The gauge transformation (5.93) becomes

$$\begin{aligned}
&|\hat{\Xi}\rangle * |\Lambda(g,\zeta)\rangle - |\Lambda(g,\zeta)\rangle * |\hat{\Xi}\rangle \\
&= e^{-\frac{1}{2}Gp^2}\Big\{ g^\mu \langle a^\dagger(\rho_1-\rho_2)(1+C)\zeta\rangle + \frac{1}{\kappa+1}\, g^\mu \langle a_\mu^\dagger\,(|\xi\rangle\langle\xi| - |C\xi\rangle\langle C\xi|)\,\frac{1}{1-T}|(1+C)\zeta\rangle \\
&\quad - 2\beta\,(p\cdot g)\left[\langle\xi|\frac{T}{1-T^2}|(1+C)\zeta\rangle - \kappa\,\langle\xi|\frac{1}{1-T^2}|(1+C)\zeta\rangle\right]\Big\}\,|\hat{\varphi}_t(\mathbf{t},p)\rangle
\end{aligned} \tag{5.95}$$

Now suppose that $\rho_2\zeta = \zeta$ and $\rho_1\zeta = 0$. We get

$$\begin{aligned}
&|\hat{\Xi}\rangle * |\Lambda(g,\zeta)\rangle - |\Lambda(g,\zeta)\rangle * |\hat{\Xi}\rangle \\
&= e^{-\frac{1}{2}Gp^2}\left[-g^\mu\langle a^\dagger(1-C)\zeta\rangle - 2\,\beta\,(p\cdot g)\,\langle\xi|\frac{T-\kappa}{1-T^2}|\zeta\rangle\right]|\hat{\varphi}_t(\mathbf{t},p)\rangle
\end{aligned} \tag{5.96}$$

Comparing now this with eq.(5.40) we see that, if we choose the ζ's in the two equations to be the same, we set $g^\mu = d^\mu$ and suitably normalize $\Lambda(g,\zeta)$, the gauge transformation (5.96) gives back just the vector state eigenfunction (5.40), or, in other words, the latter belongs to the trivial cohomology class.

Therefore, if $\zeta(k)$ is a regular function for $k \sim 0$ (henceforth let us refer to such a $|\zeta\rangle$ as *regular* or *smooth* at $k = 0$) , the vector state we have constructed in section 5.2 is cohomologically trivial. In order to get something nontrivial we have to probe the string midpoint. Therefore let us try with $|\zeta\rangle \sim C|\eta\rangle$ (from now on let us refer to the latter as *singular* or *concentrated at* $k = 0$). It satisfies $\rho_2\zeta = \zeta$ and $\rho_1\zeta = 0$ and $|(1+C)\zeta\rangle \sim |\eta_+\rangle$, $|(1-C)\zeta\rangle \sim |\eta_-\rangle$ (see eqs.(5.49)). Therefore, in this case too, as long as the parameter η remains finite, the vector state is trivial. One may be tempted to conclude that also in the limit $\eta \to 0$, such a conclusion persist and therefore the vector wavefunction we have defined be always trivial. But this would be a sloppy deduction. For in the process of taking the limit $\eta \to 0$ there emerges the true nature of cohomology.

For a cohomological problem to be well defined it is not enough to have a nilpotent operator, one must also define the set of objects which such an operator acts upon, i.e. the space of cochains. In our case a precise definitions of the cochain space has not been given so far, and it is time to fill in this gap. It is clear that the issue here is the distinction between the states that vanish and those that do not vanish in the limit $\eta \to 0$. For instance, (see (5.52,5.53)), $|0_+\rangle$ belongs to the former set (let us call it an *evanescent* state) while $|0_-\rangle$ belongs to the latter. We define the space of nonzero cochains as the space of states that are finite in the limit $\eta \to 0$, while the zero cochain is represented by 0. All this is well–defined and makes up a linear space and it is the only sensible choice to define a cohomology in this context (see Appendix D for a discussion of this point).

With the previous definition let us return to the vector eigenfunction. Thanks to the discussion following eqs.(5.95,5.96), we see immediately that if ζ in (5.40) is smooth near $k = 0$, then the corresponding wavefunction is a coboundary. If, on the other hand, $\zeta \sim |C\eta\rangle$, i.e. is concentrated at $k = 0$, then the state is a nontrivial cocycle, because we cannot figure out any non–evanescent Λ which generate it via (5.93): the only one that does the job is evanescent. This same conclusion can be drawn for the level 2 and level 3 states we found above (which were formulated directly in terms of vectors concentrated at $k = 0$). All these states are cohomologically nontrivial.

At this point we can discuss also the implication of the simplifying assumption we introduced in section 7.2 and 7.3, i.e. that the dressing function $\xi(k)$ is non-vanishing only from a certain finite negative point down to $-\infty$ in the k-axis. This assumption induced remarkable simplifications in our analysis, but that was the only reason why it was introduced: one can do without it. Anyhow let us ask ourselves what would have happened had we introduced this assumption in the vector case. In the case of ζ being concentrated at $k=0$ the last two terms at the RHS of (5.41) would vanish and we would not need to impose the transversality condition (5.42). If, on the other hand, ζ is smooth at $k=0$ then, in order to satisfy the LEOM, we would have to impose the transversality condition (5.42) together with the additional condition (5.43), but in this case we would get a trivial solution. This conclusion seems to be paradoxical only if we forget the relation between cohomology and Virasoro conditions. In fact it is perfectly logical. First of all we should remember that we have two ways of expressing the physicality of a given state. Either we say that this state is a nontrivial cocycle defined up to generic coboundaries (this is the cohomological way of putting it), or we impose conditions on the parameters of the state (polarizations) in such a way that its indeterminacy (coboundaries) get suppressed (and this is the gauge fixing way). Now, the above apparent paradox means that the simplifying assumption, which seems to suppress the transversality condition on the nontrivial cocycle (singular ζ), can be made up for by adding to the solution a trivial cocycle (regular ζ). In other words, the simplifying assumption corresponds to partially fixing the gauge freedom. It can be seen that this is true also in the more complicated cases of level 2 and level 3.

With this remarks we end our analysis of cohomology in VSFT. This problem would deserve of course a more thorough treatment, but we believe we have caught some of the essential features of it.

5.7 Proliferating solutions

All the solutions to the LEOM considered so far depend on three parameters: e, α, β. As will be seen below, e has to be set equal to 1, but the other two parameters are free. We wish to show in this section that the solutions to the LEOM are even more general than this. In fact we can prove that, if $|\hat{\varphi}(\mathbf{t},p)\rangle$ is the matter part of the tachyon solution to the linearized equation of motion, i.e. a solution to (5.6), then any state of the form

$$(\langle a^\dagger_{\mu_1}\xi_\pm\rangle \dots \langle a^\dagger_{\mu_s}\xi_\pm\rangle)|\hat{\varphi}(\mathbf{t},p)\rangle \tag{5.97}$$

where $\xi_\pm = (1 \pm C)\xi$, is also a solution for any s, with the same mass as the tachyon for any random choice of the $\pm$ signs. For

$$\begin{aligned}
&(\langle a^\dagger_\mu \xi\rangle|\hat{\varphi}(\mathbf{t},p)\rangle) * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * (\langle a^\dagger_\mu \xi\rangle|\hat{\varphi}(\mathbf{t},p)\rangle) \\
&\quad = \langle a^\dagger_\mu \xi\rangle \Big[|\hat{\varphi}(\mathbf{t},p)\rangle * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * |\hat{\varphi}(\mathbf{t},p)\rangle\Big] = \langle a^\dagger_\mu \xi\rangle|\hat{\varphi}(\mathbf{t},p)\rangle
\end{aligned} \tag{5.98}$$

The derivation of the first equality is given in Appendix D. The same can be shown if we replace $\langle a^\dagger_\mu \xi\rangle$ with $\langle a^\dagger_\mu C\xi\rangle$. This proves the above claim for $s=1$. But it is evident that now we can proceed recursively by replacing in (5.98) $|\hat{\varphi}(\mathbf{t},p)\rangle$ with $\langle a^\dagger_\nu \xi\rangle|\hat{\varphi}(\mathbf{t},p)\rangle$ and $\langle a^\dagger_\nu C\xi\rangle|\hat{\varphi}(\mathbf{t},p)\rangle$, respectively, which are also solutions, thereby proving the statement for $s=2$, and so on.

We refer to all these states as the descendants of $|\hat{\varphi}(\mathbf{t},p)\rangle$, or *tachyon descendants*. We can easily define a generating state for them

$$|\hat{\varphi}(g,\mathbf{t},p)\rangle = e^{(g_+^\mu \langle a_\mu^\dagger \xi_+\rangle + g_-^\mu \langle a_\mu^\dagger \xi_-\rangle)}|\hat{\varphi}(\mathbf{t},p)\rangle \tag{5.99}$$

By differentiating with respect to $g_\pm^\mu$ we can generate all the solutions of the type 5.97.

A similar result holds also for the other (tensor) solutions of the LEOM. At level n such states take the form

$$|\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle \equiv |\hat{\varphi}(\theta_1,...,\theta_n,\mathbf{t},p)\rangle = \sum_{i=1}^{n} \theta_i^{\mu_1\ldots\mu_i}\langle a_{\mu_1}^\dagger \zeta_1^{(i)}\rangle \ldots \langle a_{\mu_i}^\dagger \zeta_i^{(i)}\rangle |\hat{\varphi}(\mathbf{t},p)\rangle \tag{5.100}$$

where the polarizations θ_i must satisfy constraints similar to those found for level 1,2 and 3. As shown in Appendix D, we have a result similar to the above. The LEOM is satisfied with the same mass

$$\begin{aligned}&(h^\nu \langle a_\nu^\dagger \xi\rangle |\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle) * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * (h^\nu \langle a_\nu^\dagger \xi\rangle |\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle) \\ &= h^\nu \langle a_\nu^\dagger \xi\rangle \Big[|\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * |n,\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle\Big] = h^\nu \langle a_\nu^\dagger \xi\rangle |\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle\end{aligned} \tag{5.101}$$

but, now, under some conditions: either

$$\langle \xi | \frac{T-\kappa}{1-T^2} | C\zeta_j^{(i)}\rangle = 0 \tag{5.102}$$

(which is the case for instance when $\rho_2\zeta_j^{(i)} = \zeta_j^{(i)}$, $\rho_1\zeta_j^{(i)} = 0$) or, if this is not true (as is the case in our previous analysis), the polarization h is transverse to the θ_i's when contracted with the index μ_j:

$$h^\nu\, \eta_{\nu\mu_j}\, \theta_i^{\mu_1\ldots\mu_j\ldots\mu_i} = 0, \tag{5.103}$$

and this must hold $\forall i,j, \quad 1 \le j \le i, \quad 1 \le i \le n.$

Also here we can replace $\langle a_\nu^\dagger \xi\rangle$ with $\langle a_\nu^\dagger C\xi\rangle$ and obtain a new solution with the same mass, and therefore we can define the $\pm$ combination, as above. Inductively we can prove that

$$(h_1^{\nu_1}\langle a_{\nu_1}^\dagger \xi\rangle \ldots h_s^{\nu_s}\langle a_{\nu_s}^\dagger \xi\rangle)|\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle \tag{5.104}$$

satisfy the LEOM with the same mass provided each h_j is transverse to each θ_i on all indices. We can then introduce C in every $\langle a^\dagger|\xi\rangle$ factor and obtain new independent solutions. It is evident that the most general state with the same mass takes the form

$$\langle a_{\nu_1}^\dagger \xi_\pm\rangle \ldots \langle a_{\nu_s}^\dagger \xi_\pm\rangle \sum_{i=1}^{n} \theta_i^{\nu_1\ldots\nu_s;\mu_1\ldots\mu_i}\langle a_{\mu_1}^\dagger \zeta_1^{(i)}\rangle \ldots \langle a_{\mu_i}^\dagger \zeta_i^{(i)}\rangle |\hat{\varphi}(\mathbf{t},p)\rangle \tag{5.105}$$

with generic s, provided the tensor θ_i are traceless when any index ν is contracted with any index μ. However, any state of the type (5.105) is a finite linear combination of states of type (5.100). A generating function for the latter is

$$|\hat{\varphi}(h,\mathbf{t},p)\rangle = e^{(h_+^\mu \langle a_\mu^\dagger \xi_+\rangle + h_-^\mu \langle a_\mu^\dagger \xi_-\rangle)}|\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle \tag{5.106}$$

Differentiating with respect to $h_\pm$ the required number of times, we can construct any state of the type (5.100). A generating function is particularly useful in computing norms or amplitudes.

To finish this section a comment is in order concerning the enormous proliferation of solutions to the linearized equations of motion. All the states we have found seem to be cohomologically nontrivial on the basis of the analysis in the previous section. The existence of an infinite tower of descendants of a given solution is, generically speaking, hardly a surprise. We notice that a similar phenomenon is familiar in field theory. If $\phi_0(x)$ is a solution to the Klein–Gordon equation $(\partial^2 + m^2)\phi = 0$, then all the derivatives of ϕ_0 are solutions with the same mass. We conjecture that here we are coming across something similar, although the difference among different states of each tower is given here not by the application of the space derivatives (i.e. by powers of $\hat{p}$), but rather by the application of the creation operators $a_n^\dagger$, $n > 0$.

But now, the important question is: what is the nature of these states? They seem to be physical, so it is important to clarify whether they are simple copies of the first state of the tower (the *parent* state, not containing $\langle a^\dagger \xi \rangle$ factors in their $\mathcal{P}$ polynomial) or have a different physical meaning. Looking at the generating state (5.99) one can see that, if $g_\pm \sim p$, this turns into a redefinition of the arbitrary constants α and β (see section 4.1). Therefore, since these constants do not enter into physical quantities, such as G, (they might appear in quantities like H, see below, which is not by itself physical) we conclude that the states of this type are copies of the tachyon eigenfunction, without any physical differentiation from it. It is possible to see that this is true for any other tower of solutions. So the proliferation we find seems to be a proliferation of representatives of physical states (much in the same way as in the Coulomb representation of CFT we have two representatives for any vertex). This redundancy of representatives, which, it should be stressed, is due to dressing, may be a residue of the gauge symmetry of VSFT.

5.8 On the D25–brane tension

One of the unsatisfying aspects of the sliver in operator formalism was the disagreement between the energy density of the classical solution and the brane tension computed via the 3-tachyon on-shell coupling. In this section we would like to show that our approach can lead to a solution of this problem.

5.8.1 3-tachyon on-shell coupling

The cubic term of the VSFT action evaluated for 3 on-shell tachyon fields should be equal to $g_T/3$, where g_T is the 3-tachyon coupling constant for the open string, i.e.,

$$g_T = \frac{1}{g_0^2} \langle \varphi_t(\mathbf{t}, p_1) \,|\, \varphi_t(\mathbf{t}, p_2) * \varphi_t(\mathbf{t}, p_3) \rangle \Big|_{p_1^2 = p_2^2 = p_3^2 = -m_t^2 = 1} \qquad (5.107)$$

Here $|\varphi_t(\mathbf{t},p)\rangle$ must be normalized so as to give the canonical kinetic term in the low-energy action (see [23], Sec. 5.2). Using (5.38), an explicit calculation gives

$$g_T^2 = \frac{8g_0^2}{G^3} A^{13} \tilde{A}^{-1} \exp(-6H) \tag{5.108}$$

where (see Appendix D)

$$H = H_0 - \frac{(f_e-1)^2(\kappa+f_e)^2}{2(f_e+1)(f_e^3-1)} \left[\left(\frac{1}{\kappa+f_e} - \alpha \right)^2 \langle \mathbf{t}_0 | \frac{1}{1+T} | \xi \rangle^2 - \beta^2 \right] \tag{5.109}$$

and

$$A = \frac{[\det(1-\hat{\mathcal{T}}_{e_1}\hat{\mathcal{T}}_{e_2})]^3}{[\det(1-\hat{\mathcal{T}}_{e_1e_2e_3}\mathcal{M}_3)]^2} = \frac{(f_1f_2-1)^6}{(f_1f_2f_3-1)^4} \frac{\left[\det(1-T^2)\right]^3}{\left[\det(1-T\mathcal{M}_3)\right]^2} \tag{5.110}$$

$\tilde{A}$ is obtained from A by replacing all the relevant objects with tilded ones (ghost part). H_0 is a naively vanishing 'bare' term. However in level truncation it turns out to be nonvanishing due to the so–called 'twist anomaly' [61, 62].

It was shown by Okuyama that the ratio of determinants in the RHS of (5.110) diverges like $L^{5/18}$ as $L \to \infty$. Similarly, the corresponding term in $\tilde{A}$ behaves as $L^{11/18}$. Now, in order for g_T to be finite, the only possibility is to tune the "dressing" parameter e to the value 1 in some suitable way. This is the reason why, as anticipated many times in the previous sections, we have to set $e = 1$. But in the formula (5.110) this has to be done with an appropriate scaling of e to 1, in such a way as to get an overall finite result. This is very close to what we did in chapter 4 to make the dressed sliver action finite. Following the same prescription, we render separately finite A and $\tilde{A}$ (the matter and ghost part). This entails that H must be finite too. It is easy to see that the only way to implement this is to let $f_e \to 1$ (i.e. $e \to 1$) in such a way that

$$f_e - 1 = s_t L^{-5/36} \qquad \text{and} \qquad f_{\tilde{e}} - 1 = \tilde{s}_t L^{-11/36} \tag{5.111}$$

where s_t and $\tilde{s}_t$ are constants. We note that f_e and $f_{\tilde{e}}$ scale the same way as f_ϵ and $f_{\tilde{\epsilon}}$ in chapter 4.

Using $f_e \to 1$ in (5.109) we obtain $H = H_0$. From (5.108) it then follows that g_T is independent of the dressing parameters α and β. We expect this to be true for all physical quantities.

As in the case of the energy of the dressed sliver, the precise value of g_T depends not only on the value of the (so far undetermined) scaling parameter s_t, but also on the way in which the multiple limit $f_1, f_2, f_3 \to 1$ is taken. Now we would like to argue that, with the proper choice of limit prescriptions, two problems, which affect the approach with the standard sliver, may be solved:

- Validity of EOM and LEOM when contracted with the solutions themselves.
- Correct value of the product of the sliver energy density and g_T^2.

5.8.2 Scaling limit

In general observables contain such terms as $(f_1 f_2 - 1)$ and/or $(f_1 f_2 f_3 - 1)$. In the scaling limit $f_i - 1 \approx s_i L^x$, where $x < 0$ and $L \to \infty$, one expects

$$(f_1 f_2 - 1) \;\approx\; s_{12} L^x \;, \qquad (f_1 f_2 f_3 - 1) \;\approx\; s_{123} L^x \tag{5.112}$$

but the scaling coefficients s_{12} and s_{123} are a priori not unique. They depend on the precise prescription for taking the multiple limits (see Appendix C).

In chapter 4 it was shown that there is a connection between the prescription for taking limits and the validity of the EOM. Considering the EOM for the dressed sliver contracted with the dressed sliver, we have

$$\langle\, \hat{\Xi}_{\epsilon_1 \tilde{\epsilon}_1} | \mathcal{Q} \,|\, \hat{\Xi}_{\epsilon_2 \tilde{\epsilon}_2} \rangle \;=\; \left(1 - \frac{1}{f_1 f_2}\right)^{-26} \left(1 - \frac{1}{\tilde{f}_1 \tilde{f}_2}\right)^{2} \langle\, \Xi | \mathcal{Q} | \,\Xi \rangle \tag{5.113}$$

$$\langle\, \hat{\Xi}_{\epsilon_1 \tilde{\epsilon}_1} |\, \hat{\Xi}_{\epsilon_2 \tilde{\epsilon}_2} * \hat{\Xi}_{\epsilon_3 \tilde{\epsilon}_3} \rangle \;=\; \left(1 - \frac{1}{f_1 f_2 f_3}\right)^{-26} \left(1 - \frac{1}{\tilde{f}_1 \tilde{f}_2 \tilde{f}_3}\right)^{2} \langle\, \Xi |\, \Xi * \Xi \rangle \tag{5.114}$$

where $|\Xi\rangle = |\hat{\Xi}_0\rangle$ is Hata and Kawano's sliver. Let us denote

$$\zeta_{cc} \;=\; -\frac{\langle\, \Xi | \mathcal{Q} \,|\, \Xi \rangle}{\langle\, \Xi |\, \Xi * \Xi \rangle} \tag{5.115}$$

If the EOM holds for this sliver solution one gets $\zeta_{cc} = 1$. However, it was argued in [78] that this may not be the case in the level truncation regularization. We believe that this 'anomaly' should be resolved within the level truncation scheme and we expect (see below) that the result should be $\zeta_{cc} = 1$. However we would like to point out that the formalism we have presented here can allow also for values of $\zeta_{cc} \neq 1$. So, to keep this possibility into account, we leave ζ_{cc} generic. In fact, as we will see, this variable can be absorbed by the dressing.

From the requirement that 'contracted' EOM be satisfied

$$\lim_{\epsilon_i, \tilde{\epsilon}_j \to 1} \langle\, \hat{\Xi}_{\epsilon_1 \tilde{\epsilon}_1} | \mathcal{Q} \,|\, \hat{\Xi}_{\epsilon_2 \tilde{\epsilon}_2} \rangle \;=\; -\lim_{\epsilon_i, \tilde{\epsilon}_j \to 1} \langle\, \hat{\Xi}_{\epsilon_1 \tilde{\epsilon}_1} |\, \hat{\Xi}_{\epsilon_2 \tilde{\epsilon}_2} * \hat{\Xi}_{\epsilon_3 \tilde{\epsilon}_3} \rangle \tag{5.116}$$

we obtain the following condition on the scaling parameters

$$\left(\frac{s_{ccc}}{s_{cc}}\right)^{-26} \left(\frac{\tilde{s}_{ccc}}{\tilde{s}_{cc}}\right)^{2} \;=\; \zeta_{cc} \;. \tag{5.117}$$

We see that a possible anomaly in the contracted EOM can be cured by an appropriate limit prescription. However it should be noticed that the limit prescription to be used in such a case is not a priori clear and far from simply describable. We recall that in chapter 4 we defined a privileged way of taking this kind of limits: the nested limits prescription. This looked as the most natural prescription. Any other way seems to be artificial. This is the reason why we tend to believe that there should not be any anomalous ζ_{cc}.

In the case of the 'contracted' LEOM for our tachyon solution

$$\langle\hat{\phi}_e(\mathbf{t},p)|\mathcal{Q}_0|\hat{\phi}_e(\mathbf{t},p)\rangle \;=\; 0 \tag{5.118}$$

the possible anomaly [78, 27] is cured by taking

$$\zeta_{tt} \equiv -\frac{\langle\phi_t|\mathcal{Q}|\phi_t\rangle}{2\langle\phi_t|\phi_t * \Xi\rangle} = \left(\frac{s_{ttc}}{s_{tt}}\right)^{-26}\left(\frac{\tilde{s}_{ttc}}{\tilde{s}_{tt}}\right)^{2} \tag{5.119}$$

where ϕ_t is the undressed tachyon $e=0$ (from the symmetry of 3-string vertex for cyclic permutations it follows $s_{ttc}=s_{tct}=s_{ctt}$).

5.8.3 D25-brane energy

Let us now calculate the product of the dressed sliver energy density and g_T^2, which if our dressed sliver represents the D25-brane, should be

$$\left(E_c\, g_T^2\right)_{OST} \;=\; \frac{1}{2\pi^2} \tag{5.120}$$

From (5.113) and (5.108) we obtain

$$E_c\, g_T^2 \;=\; \left(\frac{s_{tt}}{s_{cc}}\right)^{26}\left(\frac{\tilde{s}_{tt}}{\tilde{s}_{cc}}\right)^{-2}\left(\frac{s_{ttt}}{s_{tt}}\right)^{-52}\left(\frac{\tilde{s}_{ttt}}{\tilde{s}_{tt}}\right)^{4}\left(E_c\, g_T^2\right)_0 \tag{5.121}$$

where $(E_c\, g_T^2)_0$ is the result for the standard sliver. In [62, 27] it was shown that $(E_c\, g_T^2)_0$ is given by

$$(E_c\, g_T^2)_0 \;=\; \frac{\pi^2}{3}\left(\frac{16}{27\ln 2}\right)^3 \tag{5.122}$$

which is obviously different from (5.120).

Note that scaling parameters s_{ttt} and $\tilde{s}_{ttt}$ do not appear in any LEOM and so are not affected by the analysis of the previous subsection. Therefore they can take values such that (5.120) is satisfied for the dressed sliver.

The possibility we have just pointed out is important because it removes a sort of no–go theorem, [78], that seemed to exist in the operator treatment of the sliver solution. However we should point out that there is a difference between the limiting/tuning procedure used in chapter 4 to define a finite energy density of the dressed sliver and the same procedure used here in order to obtain the matching between RHS and LHS of (5.120). In the first case the critical dimension was behind the argument we used and supported it (see previous chapter), in the latter case we have not been able to find a similar argument in favor of our tuning procedure. Without this the theory has apparently lost some of the predictability: see, for instance, (5.121) which is undetermined without knowing s_{ttt} and $\tilde{s}_{ttt}$. However we believe that such an argument should exist which relates tuning to the consistency of the whole theory (of which we have explored only a minute part).

Chapter 6

Chan–Paton factors and Higgsing

In the previous two chapters we have dealt with single D–branes solution. We have however pointed at the end of chapter 4 that multiple D–branes solutions are also easily obtainable in the context of VSFT.

This chapter is devoted to a description of open strings states living on a set of N D–branes. When the branes are coincident we encounter in the spectrum N^2 massless vectors, giving rise to a $U(N)$ gauge symmetry. This symmetry is part of the huge gauge symmetry of VSFT when one considers matter–ghost factorized gauge transformations. The Chan Paton factors arises from particular combinations of left/right excitations on the sliver, that takes the form the generalized Laguerre polynomials discovered in [67], see also [1]. This $U(N)$ structure is dynamically generated (it is an intrinsic part of a classical solution) and there is no need to add it by hand as in first quantized string theory or even in usual OSFT. In this sense background independence is manifest.
Using the translation operator $\mathrm{e}^{ix\hat{p}}$ we construct an array of D24–branes and analyze its small on shell fluctuations. We show that open strings stretched between parallel branes at different positions are obtained by translating differently the left and right part of the classical solution. This is possible because the lump projector is left/right factorized. Of course this operation is ambiguous for what concerns the midpoint, since it does not have a left/right decomposition. Indeed we show that a naive use of left/right orthogonality cannot give rise to the correct shift in the mass formula, proportional to the *distance*^{2} between two D–branes. By using wedge–state regularization we show that in the sliver limit there is a non vanishing contribution which is completely localized at the midpoint and gives rise to the correct shift in the mass formula. The mechanism is that of a twist anomaly, [61], which has proven to be crucial for obtaining the spectrum of strings around a single D25–brane and to give the correct ratio of D–branes. We will see at the end of the chapter in which way a dynamical change in boundary conditions is generated at the midpoint. For the sake of simplicity everything is done on the sliver state *without* the dressing deformation, for this reason issues related to overall normalizations and energy are not discussed as they are simple generalizations of the topics discussed in the previous

two chapters.

6.1 N coincident D25-branes

There are several ways to construct coincident branes solutions in VSFT, the one we are going to use is in terms of Laguerre polynomials, explicitly given in [67].
Consider a left string vector ξ_n^μ, such that

$$\rho_R \xi^\mu = 0 \tag{6.1}$$

$$\rho_L \xi^\mu = \xi^\mu \tag{6.2}$$

The $\rho_{R,L}$ operators project into the the right/left Hilbert space at zero momentum, see previous chapters[1].

With this half string vector it is possible to excite left-right symmetrically a string configuration, using the operator

$$\mathbf{x} = (a_\mu^\dagger, \xi^\mu)(a_\nu^\dagger, C\xi^\nu) = y\tilde{y} \tag{6.3}$$

where $(\cdot, \cdot)$ means inner product in level space and the operators $\tilde{y}$ y are identified with right/left excitations. The half string vector ξ is normalized by the following condition and definition

$$(\xi_\mu, \frac{1}{1-T^2}\xi^\mu) = 1 \tag{6.4}$$

$$(\xi_\mu, \frac{T}{1-T^2}\xi^\mu) = -\kappa \tag{6.5}$$

where $T = CS$ is the Sliver Neumann coefficient, (4.12).
For every choice of ξ satisfying 6.4, we can construct an infinite family of orthogonal projectors (D–branes) given by [67]

$$|\Lambda_n\rangle = (\kappa)^n L_n\left(\frac{\mathbf{x}}{\kappa}\right)|\Xi\rangle \tag{6.6}$$

where $L_n(x)$ is the n-th Laguerre polynomial. These states obey the remarkable properties

$$|\Lambda_n\rangle * |\Lambda_m\rangle = \delta_{nm}|\Lambda_m\rangle \tag{6.7}$$

$$\langle\Lambda_n|\Lambda_m\rangle = \delta_{nm}\langle\Xi|\Xi\rangle \tag{6.8}$$

Due to these properties, once the sliver is identified with a single D–brane, a stack of N D-branes can be given by

$$|N\rangle = \sum_{n=0}^{N-1} |\Lambda_n\rangle \tag{6.9}$$

[1]In this chapter we use the notation $\rho_L \equiv \rho_2$ and $\rho_R \equiv \rho_1$, in order to make more explicit the left/right splitting of the classical solutions we present

From (6.8) we further get that the *bpz* norm of such a solution is N–times the one of the sliver.

So far we have considered left-right symmetric projectors which are in one to one correspondence with type 0 Laguerre polynomial, there are however non left-right symmetric states corresponding to generalized Laguerre polynomials, they are given by, [1]

$$|\Lambda_{nm}\rangle = \sqrt{\frac{n!}{m!}}\kappa^m (iy)^{n-m} L_m^{n-m}\left(\frac{x}{\kappa}\right)|\Xi\rangle \quad n \geq m \tag{6.10}$$

$$|\Lambda_{nm}\rangle = \sqrt{\frac{m!}{n!}}\kappa^n (i\tilde{y})^{n-m} L_n^{m-n}\left(\frac{x}{\kappa}\right)|\Xi\rangle \quad m \geq n \tag{6.11}$$

and obey the properties [2]

$$|\Lambda_{nm}\rangle * |\Lambda_{pq}\rangle = \delta_{mp}|\Lambda_{nq}\rangle \tag{6.12}$$

$$\langle\Lambda_{nm}|\Lambda_{pq}\rangle = \delta_{mp}\delta_{nq}\langle\Xi|\Xi\rangle \tag{6.13}$$

note in particular that $|\Lambda_n\rangle = |\Lambda_{nn}\rangle$. With these states we can implement partial–isometry–like operations, see also [85]. Consider indeed

$$|-+\rangle = \sum_{n=0}^{N-1} |\Lambda_{n+1,n}\rangle \tag{6.14}$$

$$|+-\rangle = \sum_{n=0}^{N-1} |\Lambda_{n,n+1}\rangle \tag{6.15}$$

It's trivial to see that

$$|+-\rangle * |-+\rangle = |N\rangle \tag{6.16}$$

$$|-+\rangle * |+-\rangle = |N\rangle - |\Xi\rangle \tag{6.17}$$

Note that any of the previous states can be obtained starting from the sliver by star products

$$|\Lambda_{nm}\rangle = (|-+\rangle)_*^n * |\Xi\rangle * (|+-\rangle)_*^m \tag{6.18}$$

We have in particular

$$\begin{aligned} |+-\rangle * |\Xi\rangle &= 0 \\ |\Xi\rangle * |-+\rangle &= 0 \end{aligned} \tag{6.19}$$

As a final remark it is worth noting that the partial isometry that relates projectors to projectors is actually a $*$–rotation and hence a (matter ghost factorized) gauge transformation. We have indeed

$$\Lambda_{nn} = \mathrm{e}^{\frac{\pi}{2}(\Lambda_{nm}-\Lambda_{mn})}\,\Lambda_{mm}\,\mathrm{e}^{-\frac{\pi}{2}(\Lambda_{nm}-\Lambda_{mn})} \tag{6.20}$$

as can be easily checked from (6.12)

[2] Another realization of this algebra is given in [72]

6.2 $U(N)$ open strings

Let's recall that the (matter–ghost factorized) open string cohomology around a (matter–ghost factorized) classical solution $|\Psi\rangle$ is given by the following conditions

$$|\phi\rangle = |\phi\rangle * |\Psi\rangle + |\Psi\rangle * |\phi\rangle \tag{6.21}$$

$$|\phi\rangle \neq |\Lambda\rangle * |\Psi\rangle - |\Psi\rangle * |\Lambda\rangle \tag{6.22}$$

The first representing $\mathcal{Q}_\Psi$–closed states while the second gauges away $\mathcal{Q}_\Psi$ exact–states.[3] In the case of N–coincident D25–branes the classical solution is given by (6.9).

As multiple D–branes are obtained starting from the sliver by multiple $*$–products via (6.18), a generic open string state on the sliver can acquire a Chan–Paton factor $(i,j) \in Adj[U(N)]$ in the same way.
Let $|\{g\}, p\rangle$ be an on–shell open string state on the sliver, identified by the collection of polarization tensors $\{g\}$ and momentum p. The Chan–Paton structure is simply given by

$$|(i,j);\{g\},p\rangle = (|-+\rangle)_*^i * |\{g\},p\rangle * (|+-\rangle)_*^j \tag{6.23}$$

There is a subtlety here, related to twist anomaly, [61], and the consequent breakdown of $*$–associativity. Indeed the expression (6.23) is ambiguous in the overall normalization in front: it depends on how the various star products involved are nested. This is so because all the states we are considering are constructed on the sliver, which fails to satisfy unambiguously its equation of motion when states at non zero momentum enter the game. Consider for simplicity the Hata–Kawano tachyon state, [23, 86]

$$|p\rangle = \mathcal{N} \mathrm{e}^{(-ta^\dagger + ix)p} |\Xi\rangle = \mathcal{N}' \mathrm{e}^{ip\hat{X}\left(\frac{\pi}{2}\right)} |\Xi\rangle \tag{6.24}$$

$$t = 3\frac{T^2 - T + 1}{1+T} v_0 \tag{6.25}$$

this state satisfies (weakly) the linearized equation of motion (LEOM) with the sliver state

$$|p\rangle * |\Xi\rangle = |\Xi\rangle * |p\rangle = \mathrm{e}^{-Gp^2} |p\rangle \tag{6.26}$$

$$G = \log 2 \quad \Rightarrow \quad p^2 = 1 \tag{6.27}$$

The quantity G gets a non vanishing value from the region very near $k = 0$ in the continuous basis, where some of the remarkable properties, encoding associativity, between Neumann coefficients breaks down due to singularities that are regulated in a non associative way (like level truncation). Indeed (6.26) violates associativity if, as is the case, $G \neq 0$

$$(|p\rangle * |\Xi\rangle) * |\Xi\rangle \neq |p\rangle * (|\Xi\rangle * |\Xi\rangle) \tag{6.28}$$

[3] These conditions actually cover only the ghost–matter factorized cohomology

Just to fix a convention (and stressing once more that the only ambiguity is in the overall normalization) we decide to do first all the star products at zero momentum (that do not develop twist anomaly) and, as the last operation, multiply the result with the state at definite momentum $|\{g\},p\rangle$

Now we show that (6.23) satisfies the LEOM

$$|(i,j);\{g\},p\rangle = |(i,j);\{g\},p\rangle * |N\rangle + |N\rangle * |(i,j);\{g\},p\rangle \tag{6.29}$$

using (6.19) we get the relations

$$|N\rangle * (|-+\rangle)_*^i = (|-+\rangle)_*^i * |N-i\rangle \tag{6.30}$$

$$(|+-\rangle)_*^j * |N\rangle = |N-j\rangle * (|+-\rangle)_*^j \tag{6.31}$$

which allow to write the LEOM as

$$|(i,j);\{g\},p\rangle = (|-+\rangle)_*^i * \Big(|\{g\},p\rangle * |N-j\rangle + |N-i\rangle * |\{g\},p\rangle\Big) * (|+-\rangle)_*^j$$

Now we prove that

$$|\{g\},p\rangle * |\Lambda_{n\geq 1}\rangle = |\Lambda_{n\geq 1}\rangle * |\{g\},p\rangle = 0 \tag{6.32}$$

A general open string state on the sliver $|\Xi\rangle$ can be obtained differentiating the generating state, see appendix D

$$|\phi_\beta\rangle = \mathrm{e}^{-(tp+\beta)\cdot a^\dagger}|\Xi\rangle \mathrm{e}^{ipx} \tag{6.33}$$

where

$$t = 3\frac{T^2-T+1}{1+T}v_0 \tag{6.34}$$

is the on–shell tachyon vector, [23] and β_μ is a level–Lorentz vector.
The $|\Lambda_n\rangle$'s can be generated by the state ,[67]

$$|\Xi_\lambda\rangle = \mathrm{e}^{\lambda\cdot a^\dagger}|\Xi\rangle \tag{6.35}$$

General formulas of [30] and appendix D allows to compute

$$|\phi_\beta\rangle * |\Xi_\lambda\rangle \quad = \quad \mathrm{e}^{-Gp^2+A_{LR}(\beta,\lambda)}\mathrm{e}^{-(tp+\rho_L\beta-\rho_R\lambda)\cdot a^\dagger}|\Xi\rangle \mathrm{e}^{ipx} \tag{6.36}$$

$$|\Xi_\lambda\rangle * |\phi_\beta\rangle \quad = \quad \mathrm{e}^{-Gp^2+A_{RL}(\beta,\lambda)}\mathrm{e}^{-(tp+\rho_R\beta-\rho_L\lambda)\cdot a^\dagger}|\Xi\rangle \mathrm{e}^{ipx} \tag{6.37}$$

where

$$\begin{aligned} A_{LR}(\beta,\lambda) \quad = \quad & -\frac{1}{2}(\beta\cdot,\frac{T}{1-T^2}\beta) + (\beta\cdot,\frac{\rho_R-T\rho_L}{1-T^2}C\lambda) - \frac{1}{2}(\lambda\cdot,\frac{T}{1-T^2}\lambda) \\ & +p\cdot\left((t,\frac{T}{1-T^2}\beta) - (t,\frac{\rho_L-T\rho_R}{1-T^2}\lambda) + (t,\frac{\rho_R+T\rho_L}{1-T^2}\beta) - (t,\frac{\rho_L-\rho_R}{1-T^2}\lambda)\right) \end{aligned} \tag{6.38}$$

and

$$A_{RL}(\beta,\lambda) = A_{LR}(\beta,\lambda)\Big|_{\rho_L\to\rho_R,\,\rho_R\to\rho_L} \tag{6.39}$$

We can restrict the polarization β_n^μ to the $k=0$ component, indeed every physical excitation of the tachyon wave function $\mathrm{e}^{-tp\cdot a^\dagger+ipx}|\Xi\rangle$ should be localized there, see chapter 4. Therefore is not restrictive to ask

$$(\beta\cdot, f(T)\xi) = 0 \tag{6.40}$$

once the half string vector ξ^μ vanishes rapidly enough at $k=0$, see the previous two chapters for explicit realizations of this condition.
We also ask the following condition

$$\left(t, \frac{1}{1\pm T}\xi^\mu\right) \quad = \quad 0 \tag{6.41}$$

This condition states that the half string vector ξ^μ should be "orthogonal" to the on–shell tachyon vector $t = 3\frac{T^2-T+1}{1+T}v_0$, this just constrains $2D$ components of ξ^μ out of $D\infty - 1$, and as such is easy to implement. [4]

Now, using (6.12), it is easy to show that

$$|\{g\},p\rangle * |N\rangle + |N\rangle * |\{g\},p\rangle = |\{g\},p\rangle * |\Xi\rangle + |\Xi\rangle * |\{g\},p\rangle, \tag{6.42}$$

as claimed.
Given (6.32) it follows directly that

$$|\{g\},p\rangle * |N-j\rangle + |N-i\rangle * |\{g\},p\rangle = |\{g\},p\rangle * |\Xi\rangle + |\Xi\rangle * |\{g\},p\rangle \tag{6.43}$$

hence the LEOM simplifies to

$$|(i,j);\{g\},p\rangle = (|-+\rangle)_*^i * \Big(|\{g\},p\rangle * |\Xi\rangle + |\Xi\rangle * |\{g\},p\rangle\Big) * (|+-\rangle)_*^j \tag{6.44}$$

This ensures the on–shellness of the state $|(i,j)\ ;\ \{g\},p\rangle$ once this is true for $|\{g\},p\rangle$. We thus recover N^2 kinematical copies of the spectrum on a single D–brane. Note that the left/right structure of these states is the same as a $U(N)$ double line notation, as the relations (6.12) certify. It should be noted that this Chan–Paton structure does not sit at the endpoints of the string, [87], but is rather "diluted" on the string halves. This can be traced back to the singular field redefinition that should relate OSFT with VSFT, see the conclusions.

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[4] These conditions are actually not needed if we represent the tachyon state as $\mathrm{e}^{ip\hat{X}(\frac{\pi}{2})}|\Xi\rangle$ since, up to overall normalizations, midpoint insertions commutes with the star product; the role of such conditions is to avoid extra terms when we use the CBH formula to pass to the oscillator expression $\mathrm{e}^{(-ta^\dagger+ix)p}|\Xi\rangle$.

6.3 N coincident D24-branes

A system of N coincident D24–branes can be represented by[5]

$$|N\rangle = \left(\sum_{n=0}^{N} |\Lambda_n\rangle\right) \otimes |\Xi'\rangle \tag{6.45}$$

where the state $|\Xi'\rangle$ is the lump solution given in [54]. The open string string sector with Lorentz indices coming from the 25 dimensional world volume (N^2 tachyons, $U(N)$–gluons, etc...) is exactly as in the previous section. In addition there are the physical states coming from transverse excitations. These states are given by exciting the transverse part of the classical solution $|\Xi'\rangle$ with oscillators. The Chan–Paton degrees of freedom are encoded in the worldvolume part of the state as in the previous section. For example the transverse scalars are given by[6]

$$|(ij); g^{25}, p_{\|}\rangle = \mathcal{N}\, \mathrm{e}^{(-ta^\dagger + ix)p_{\|}} |\Lambda_{ij}\rangle \otimes g^{25} \cdot a'^{\dagger}_{25} |\Xi'\rangle \tag{6.46}$$

It's easy to verify that these states satisfy the LEOM iff $p_{\|}^2 = 0$, we have indeed

$$\begin{aligned} |(ij); g^{25}, p_{\|}\rangle &= |N\rangle * |(ij); g^{25}, p_{\|}\rangle + |(ij); g^{25}, p_{\|}\rangle * |N\rangle && (6.47)\\ &= 2^{-p_{\|}^2} \mathcal{N}\, \mathrm{e}^{(-ta^\dagger + ix)p_{\|}} |\Lambda_{ij}\rangle \otimes (\rho'_L + \rho'_R) g^{25} \cdot a'^{\dagger}_{25} |\Xi'\rangle && (6.48) \end{aligned}$$

The $\rho'_{L,R}$ are the left/right projectors with zero modes, see [30].

The level vector g^{25} is completely arbitrary, but only its midpoint part is not pure gauge. Indeed, exactly as in [83], we can try to gauge away any of the states (6.46)

$$\delta |(ij); g^{25}, p_{\|}\rangle = |Q_{ij}\rangle * |N\rangle - |N\rangle * |Q_{ij}\rangle \tag{6.49}$$

where

$$|Q_{ij}\rangle = -\mathrm{e}^{(-ta^\dagger + ix)p_{\|}} |\Lambda_{ij}\rangle \otimes u^{25} \cdot a'^{\dagger}_{25} |\Xi'\rangle \tag{6.50}$$

We have

$$\delta |(ij); g^{25}, p_{\|}\rangle = -|(ij); (\rho'_L - \rho'_R) u^{25}, p_{\|}\rangle \tag{6.51}$$

Thus the state is pure gauge if

$$g^{25} = (\rho'_L - \rho'_R) u^{25} \tag{6.52}$$

It is well known that the operator $(\rho'_L - \rho'_R)$ just change the twist parity of a given level vector. In the diagonal basis all vectors are paired except the one corresponding to $k = 0$

[5] Other possibilities, for example putting the Laguerre polynomials on the codimension, are related to this by partial isometry and hence, due to (6.20), should be gauge equivalent

[6] Note that once the relations (6.41) are implemented one can recast the Chan Paton indices directly on the classical solution and then act with oscillators to build onshell fluctuation

that is only twist even, [50] (at least if we restrict ourselves to vectors that have a non vanishing overlap with Fock–space vectors, see chapter 5). Thus the gauge transformation (6.51) gauges away all the components of g^{25} except the $k=0$ one, which is the midpoint. One can construct higher transverse excitations by applying more and more transverse oscillators as in the previous chapter. Again only the $k=0$ oscillator(s) are not gauge trivial.

6.4 Higgsing

Now we want to "higgs" the previous system of N coincident D24–branes to an array of N D24–branes, displaced of a distance ℓ from one another in the transverse dimension y. This system is obtained by multiple translation of the previous classical solution (6.45).

$$|N^{(\ell)}\rangle = \sum_{n=0}^{N-1}\left(|\Lambda_n\rangle \otimes \mathrm{e}^{-in\ell\hat{p}}|\Xi'\rangle\right) \tag{6.53}$$

As in [54] it is very convenient to pass to the oscillator basis by

$$\hat{p} = \frac{1}{\sqrt{b}}\left(a_0 + a_0^\dagger\right) \tag{6.54}$$

and to define the level vector

$$\beta_N = -\frac{i\ell}{\sqrt{b}}\left(1-T'\right)_{0N} \tag{6.55}$$

The transverse part of the n-th D24–brane in (6.53) can thus be written as

$$|\Xi'_n\rangle = \mathrm{e}^{in\ell\hat{p}}|\Xi'\rangle = \mathrm{e}^{\frac{n^2}{2}\left(\beta,\frac{1}{1-T'}\beta\right)+n\beta\cdot a'^\dagger}|\Xi'\rangle \tag{6.56}$$

As proven in the next chapter we have

$$|\Xi'_n\rangle * |\Xi'_m\rangle = \delta_{nm}|\Xi'_n\rangle \tag{6.57}$$

$$\langle\Xi'_n|\Xi'_m\rangle = \delta_{nm}\langle\Xi'|\Xi'\rangle \tag{6.58}$$

We recall here that the orthogonality condition comes from a divergence at $k=0$ of the continuous basis of the primed Neumann matrices. Indeed, up to unimportant contributions, we have used the identification, see next chapter

$$\delta_{nm} = \exp\left[(n-m)^2\left(\beta,\frac{1}{1+T'}\beta\right)\right] = \exp\left[-\frac{\ell^2}{b}(n-m)^2\left(\frac{(1-T')^2}{1+T'}\right)_{00}\right] \tag{6.59}$$

Note that we don't really need to use different projectors on the worldvolume as the degeneracy is lifted by the different space–translations of the various projectors, however one can still use the $|\Lambda_n\rangle$'s in order to maintain the orthogonality as $\ell \to 0$.

Now we come to the spectrum.

Type (n,n) strings (the ones stretched between the same D–brane) are obtained by translation of strings on a single D24–brane

$$|(n,n);\{g\},p_{\|}\rangle = \mathrm{e}^{in\ell p_{\perp}}|\{g\},p_{\|}\rangle^{(n)} \tag{6.60}$$

where $|\{g\},p_{\|}\rangle^{(n)}$ is an on–shell state of the previous section constructed on $|\Lambda_{nn}\rangle \otimes |\Xi'\rangle$. Thus we get N copies of the spectrum of a single D24–branes: N tachyons, N massless vectors etc... This gives a $U(1)^N$ gauge symmetry.

The situation changes when we want to consider strings stretched between two different D–branes. In this case we expect that a shift in the mass formula is generated, proportional to the square of the distance between the two branes. In order to construct (i,j) states we have to translate the state $|\Xi'\rangle$ differently with respect its left/right degrees of freedom. We use the following identification for the left/right momentum

$$\hat{p} = \hat{p}_L + \hat{p}_R \tag{6.61}$$

$$\hat{p}_{L,R} = \frac{1}{\sqrt{b}}\left(\rho'_{L,R}a + \rho'_{L,R}a^{\dagger}\right)_0 \tag{6.62}$$

We then consider the state

$$\mathrm{e}^{-in\ell\hat{p}_L - im\ell\hat{p}_R}|\Xi'\rangle \propto \mathrm{e}^{(n\beta_L+m\beta_R)\cdot a'^{\dagger}}|\Xi'\rangle = |\Xi'_{nm}\rangle \tag{6.63}$$

where we have defined

$$\beta_{L,R} = \rho'_{L,R}\beta \tag{6.64}$$

The ρ' projectors obey the following properties up to midpoint subtleties, see later

$$\rho'_L + \rho'_R = 1 \tag{6.65}$$

$$(\rho'_{L,R})^2 = \rho'_{L,R} \tag{6.66}$$

$$\rho'_{L,R}\rho'_{R,L} = 0 \tag{6.67}$$

If we naively use these properties, using the formulas of [54], it is easy to prove that

$$|\Xi'_{nm}\rangle * |\Xi'_{pq}\rangle = \mathrm{e}^{-\frac{1}{2}\left((nm+pq-nq)\beta\frac{1}{1-T'}\beta + (mp-nm-pq+nq)\beta\frac{1}{1-T'^2}\beta\right)}|\Xi'_{nq}\rangle \tag{6.68}$$

We can then normalize the above states in order to have

$$|\hat{\Xi}'_{nm}\rangle * |\hat{\Xi}'_{pq}\rangle = \delta_{mp}|\hat{\Xi}'_{nq}\rangle \tag{6.69}$$

where

$$|\hat{\Xi}'_{nm}\rangle = \mathrm{e}^{\frac{1}{4}\beta\frac{n^2+m^2+2nmT'}{1-T'^2}\beta}\mathrm{e}^{(n\beta_L+m\beta_R)\cdot a'^{\dagger}}|\Xi'\rangle \tag{6.70}$$

Note that this normalization is quite formal as the quantity $\beta\frac{1}{1+T}\beta$ is actually divergent, this is not however a real problem as open string states are not normalized by the LEOM's, moreover it should be noted that even if these left/right non–symmetric states have a vanishing normalization, they give rise to non vanishing objects (the projectors) by $*$ product.

Consider now, for simplicity, the "tachyon" state stretched from the i-th brane to the j-th. The corresponding state is given by

$$|(ij); p_{\|}\rangle = \mathcal{N} \mathrm{e}^{(-ta^\dagger + ix)p_{\|}} |\Xi\rangle \otimes |\hat{\Xi}'_{ij}\rangle \tag{6.71}$$

Using (6.69) it is easy to see that the above state satisfies the LEOM

$$|(ij)\ ;\ p_{\|}\rangle = |(ij)\ ;\ p_{\|}\rangle * |N^{(\ell)}\rangle + |N^{(\ell)}\rangle * |(ij)\ ;\ p_{\|}\rangle = 2^{-p_{\|}^2+1} |(ij)\ ;\ p_{\|}\rangle \tag{6.72}$$

with $p_{\|}^2 = 1$, that is we don't get the usual mass shift proportional to the *distance*^{2} between the two D–branes.

However the algebra (6.69) is not quite correct. To elucidate this point it is worth considering the components of the level vector β in the continuous part of the diagonal basis of the primed Neumann matrices, see appendix B for details. [7] We have

$$\beta(k) = -\frac{i\ell}{\sqrt{b}} (1 + \mathrm{e}^{-\frac{\pi |k|}{2}}) V_0(k) \tag{6.73}$$

where $V_0(k)$ is the zero component of the normalized eigenvector of the continuous basis,

$$V_0(k) = \sqrt{\frac{bk}{4 \sinh \frac{\pi k}{2}}} \left[4 + k^2 \left(\Re F_c(k) - \frac{b}{4} \right)^2 \right]^{-\frac{1}{2}} \tag{6.74}$$

The β vector is finite at $k = 0$,

$$\beta(0) = -\frac{i\ell}{\sqrt{2\pi}}, \tag{6.75}$$

hence its left/right decomposition is not well defined. This implies that it is not correct to consider the quantity

$$\gamma = \left(\beta_L, \frac{1}{1+T'} \beta_R \right) = -\frac{\ell^2}{b} \int_{-\infty}^{\infty} \theta(k)\theta(-k) \frac{\left(1 + \mathrm{e}^{-\frac{\pi|k|}{2}}\right)^2}{1 - \mathrm{e}^{-\frac{\pi|k|}{2}}} V_0(k)^2 \tag{6.76}$$

as vanishing since it is formally indeterminate (it is "$0 \cdot \infty$"). Assuming that γ is non vanishing one easily obtains that the algebra (6.69) gets modified to

$$|\hat{\Xi}'_{nm}\rangle * |\hat{\Xi}'_{pq}\rangle = \delta_{mp}\, \mathrm{e}^{\frac{1}{4}\left[(n-p)^2 + (m-q)^2\right]\gamma}\, |\hat{\Xi}'_{nq}\rangle \tag{6.77}$$

Taking this modification into account we obtain

$$|(nm)\ ;\ p_{\|}\rangle * |N\rangle + |N\rangle * |(nm)\ ;\ p_{\|}\rangle = 2^{-p_{\|}^2 + 1 + \frac{1}{4}(n-m)^2 \frac{\gamma}{\log 2}} |(nm)\ ;\ p_{\|}\rangle \tag{6.78}$$

[7] There are, of course, also the contributions from the discrete spectrum, but they are not singular for $0 < b < \infty$

that gives the mass formula

$$p_{\|}^2 = 1 + \frac{1}{4}(n-m)^2 \frac{\gamma}{\log 2} \tag{6.79}$$

We recall that the mass for such a state should be given by ($\alpha' = 1$)

$$p_{\|}^2 = 1 - \left(\frac{\Delta y_{nm}}{2\pi}\right)^2 = 1 - \left(\frac{(n-m)\ell}{2\pi}\right)^2 \tag{6.80}$$

The two formulas agrees iff

$$\gamma = \left(\beta_L, \frac{1}{1+T'}\beta_R\right) = -\frac{\ell^2}{\pi^2}\log 2 \tag{6.81}$$

To verify this identity we need to regularize the ambiguous expression (6.76). We do it by substituting the lump Neumann coefficient T' with the wedge states one T'_N. We remind that, see [72]

$$T'_N = \frac{T' + (-T')^{N-1}}{1-(-T')^N} \tag{6.82}$$

$$T'_N \star T'_N = X' + (X'_+, X'_-)(1 - \mathcal{T}'_N \mathcal{M}')^{-1} \mathcal{T}'_N \begin{pmatrix} X'_- \\ X'_+ \end{pmatrix} = T'_{2N-1} \tag{6.83}$$

We have[8]

$$\gamma = \left(\beta, \rho_L(T') \frac{1}{1+T'\star T'} \rho_R(T')\beta\right) = \lim_{N\to\infty}\left(\beta \rho_L(T'_N) \frac{1}{1+T'_{2N}} \rho_R(T'_N)\beta\right) \tag{6.84}$$

where we have used the $*$–multiplication between wedge states

$$T'_N \star T'_N = T'_{2N-1} \backsimeq T'_{2N}, \quad N \gg 1 \tag{6.85}$$

The matrices T'_N gets contributions from the continuous and the discrete spectrum but only the continuous spectrum is relevant in the large N limit, moreover it is only the region infinitesimally near the point $k = 0$ that really contributes. We have

$$\gamma = \left(\frac{i\ell}{\sqrt{2\pi}}\right)^2 \lim_{N\to\infty} \int_{-\infty}^{\infty} dk \rho_L^{(N)}(k) \frac{1}{1+t_{2N}(k)} \rho_R^{(N)}(k) \tag{6.86}$$

with

$$t_N(k) = \frac{-e^{-\frac{\pi k}{2}} + \left(e^{-\frac{\pi k}{2}}\right)^{N-1}}{1 - \left(e^{-\frac{\pi k}{2}}\right)^N} \tag{6.87}$$

$$\rho_L^{(N)}(k) = 1 - \frac{1}{1 + \left(e^{-\frac{\pi k}{2}}\right)^{N-1}} \tag{6.88}$$

$$\rho_R^{(N)}(k) = \frac{1}{1 + \left(e^{-\frac{\pi k}{2}}\right)^{N-1}} \tag{6.89}$$

[8] Note that the T' in the denominator of (6.76) is actually obtained by the projector equation $T' \star T' = T'$ that is violated in wedge–state regularization

where we have used the expression of $\rho'_{L,R}$ in terms of the sliver matrix and the $*$ Neumann coefficients, appendix A, and their (continuous) eigenvalues, appendix B. Let's evaluate the integral in the large N limit ($x = -\frac{\pi k}{2}$, $y = Nx$)

$$\begin{aligned}
&\int_{-\infty}^{\infty} dk \rho_L^{(N)}(k) \frac{1}{1+t_{2N}(k)} \rho_R^{(N)}(k) \\
= \quad & -\frac{2}{\pi} \int_{-\infty}^{\infty} dx \frac{\mathrm{e}^{Nx}\left(\mathrm{e}^{Nx}-1\right)}{\left(1-\mathrm{e}^{x}\right)\left(1+\mathrm{e}^{Nx}\right)\left(1+\mathrm{e}^{2Nx}\right)} + O\left(\frac{1}{N}\right) \\
= \quad & -\frac{2}{N\pi} \int_{-\infty}^{\infty} dy \frac{\mathrm{e}^{y}\left(\mathrm{e}^{y}-1\right)}{\left(1-\mathrm{e}^{\frac{y}{N}}\right)\left(1+\mathrm{e}^{y}\right)\left(1+\mathrm{e}^{2y}\right)} + O\left(\frac{1}{N}\right) \\
= \quad & \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{dy}{y} \frac{\mathrm{e}^{y}\left(\mathrm{e}^{y}-1\right)}{\left(1+\mathrm{e}^{y}\right)\left(1+\mathrm{e}^{2y}\right)} + O\left(\frac{1}{N}\right) \\
= \quad & \frac{2}{\pi} \log 2 + O\left(\frac{1}{N}\right)
\end{aligned} \tag{6.90}$$

so we get the right value of γ. Note that, when $N \to \infty$, the integrand completely localizes at $k = 0$, as claimed before.

As a last remark we would like to discuss about the position of the midpoint in the transverse direction for the states $|\Xi'_{nm}\rangle$. We know that for the usual lump solution we have, [64]

$$\hat{X}\left(\frac{\pi}{2}\right)|\Xi'\rangle = 0 \tag{6.91}$$

That is the lump functional has support on string states in which the midpoint is constrained to live on the worldvolume. This is interpreted as a Dirichlet condition, see also [88]. Moreover, since we have

$$\left[\hat{X}\left(\frac{\pi}{2}\right), p\right] = [x_0, p] = i, \tag{6.92}$$

it is immediate to see that

$$\hat{X}\left(\frac{\pi}{2}\right) \mathrm{e}^{-in\hat{p}\ell}|\Xi'\rangle = n\ell\, \mathrm{e}^{-in\hat{p}\ell}|\Xi'\rangle \tag{6.93}$$

So shifted branes undergoes a consistent change in boundary conditions.

The operator $\hat{X}\left(\frac{\pi}{2}\right)$ is proportional to the $k = 0$ position operator, [50]

$$\hat{X}\left(\frac{\pi}{2}\right) = 2\sqrt{\pi}\hat{x}_{k=0} \tag{6.94}$$

It's easy to check that we have the following commutation relations

$$\left[2\sqrt{\pi}\hat{x}_k, p\right] = 2i\sqrt{\frac{2\pi}{b}} V_0(k) \tag{6.95}$$

$$\left[2\sqrt{\pi}\hat{x}_k, p_L\right] = 2i\sqrt{\frac{2\pi}{b}} V_0(k)\theta(-k) \tag{6.96}$$

$$\left[2\sqrt{\pi}\hat{x}_k, p_R\right] = 2i\sqrt{\frac{2\pi}{b}} V_0(k)\theta(k) \tag{6.97}$$

That allows to write

$$\lim_{k\to 0^-} 2\sqrt{\pi}\hat{x}_k \mathrm{e}^{-i(n\hat{p}_L+m\hat{p}_R)\ell}|\Xi'\rangle = n\ell \mathrm{e}^{-i(n\hat{p}_L+m\hat{p}_R)\ell}|\Xi'\rangle \tag{6.98}$$

$$\lim_{k\to 0^+} 2\sqrt{\pi}\hat{x}_k \mathrm{e}^{-i(n\hat{p}_L+m\hat{p}_R)\ell}|\Xi'\rangle = m\ell \mathrm{e}^{-i(n\hat{p}_L+m\hat{p}_R)\ell}|\Xi'\rangle \tag{6.99}$$

The string functional relative to this state is not continuous at the midpoint, this is the reason why the correct mass shell condition comes out from a twist anomaly. In the singular representation of VSFT in which the whole interior of a string is contracted to the midpoint, [22], these properties reproduce the expected change in the left/right boundary conditions, and show that the point $k = 0$ naturally accounts for D–branes moduli.

Chapter 7

Time dependent solutions: decay of D–branes

The search for time–dependent solutions has lately become one of the prominent research topics in string theory. Particularly interesting is the search for solutions describing the decay of D-branes. An archetype problem in open bosonic string theory is describing the evolution from the maximum of the tachyon potential to the (local) minimum. Such a solution known as rolling tachyon, if it exists, describes the decay of the space filling D25–brane corresponding to the unstable perturbative vacuum to the locally stable vacuum. That such a solution exists has been argued in many ways, [76], see also [89, 90, 91]. A natural framework where to study such a nonperturbative problem is String Field Theory (SFT). But, although there have been some attempts to describe such phenomena in a SFT framework [77], no analytical control has been achieved so far.

We will see in this chapter that exact analytical solutions are easily obtained in VSFT. We will indeed show that the matter star algebra contains exact time–dependent projectors with the appropriate characteristic to represent S-branes, that is solitonic solutions localized in time. We show that the time profile of such solutions is dominated for large t by a factor $\exp(-at^2)$ with positive constant a. At time $t = 0$ the solution takes the form of a deformed sliver (D25-brane), the deformation being parameterized by two continuous parameters. At infinite future (and infinite past) time it becomes 0, i.e. it flows into the stable vacuum. If the initial configuration happens to coincide exactly with the sliver (no deformation present) there cannot be any time evolution. Therefore an initial deformation away from the sliver is essential for true time evolution. Needless to say this is strongly reminiscent of Sen's rolling tachyon solution, [76] or of an S–brane, [92], i.e of a state finely tuned to be poised at the initial time near the top of the tachyon potential and let free to evolve.

The technique to produce such a solution is based on double Wick–rotation, as is customary in such kind of trade. Our reference solution is obtained by picking a Euclidean lump solution with one transverse space direction (a D24-brane) and then performing an inverse Wick–rotation along such a direction. However the important ingredient is that our lump solution is not the ordinary one. Since the spectrum of the twisted Neumann coefficient matrices of the three strings vertex nicely split into a continuous and a discrete part, we define a new solution in which the squeezed state matrix is made of a continuous part, which is the same as for the conventional lump, and a discrete part which is *inverted* with respect to the ordinary lump. We call this *unconventional lump*, see eqs.(7.12,7.13) below. After inverse–Wick–rotating it we get the desired time behavior, (7.30).

In the previous paragraphs we have informally talked about time. Now we would like to be more precise. Our time is nothing but a Wick–rotated space coordinate, representing the position of the string center–of–mass, and it couples to the open string (flat) metric. In the conclusive section we will discuss a possible connection of such time with the closed string time (which couples to the bulk gravity metric). See [93, 94, 95, 96] for discussions related to the definition of time in SFT.

We will then show how to add an E–field to the above construction. The presence of the Kalb–Ramond field is important since fundamental strings are charged with respect to it (this is in fact the only conserved charge of bosonic string theory on topological trivial spaces). When the E–field is turned on the decay products of a D–brane contain such fundamental strings, even at the tachyon vacuum. A description of them in VSFT is given in the last section of this chapter; as expected they can be properly defined only in a $B_{\mu\nu}$ background.

7.1 Time dependent solutions: dead ends

In order to appreciate the very nature of the problem of finding time–localized VSFT solutions, let us examine first some obvious attempts and learn from their failure. The first thing that comes to one's mind is to start from a lump with one transverse space direction (therefore it represents a D24-brane) and inverse–Wick–rotate it. For simplicity we denote the transverse direction coordinate, momentum and oscillators simply by x, p and a_N. The solution is written as follows:

$$\begin{aligned} |\Psi'\rangle &= |\Xi\rangle_{25} \otimes |\Lambda'\rangle \\ |\Lambda'\rangle &= \mathcal{N}' \exp\left[-\frac{1}{2}\sum_{N,M\geq 0} a_N^\dagger S'_{NM} a_M^\dagger\right]|\Omega_b\rangle \end{aligned} \tag{7.1}$$

where $|\Xi\rangle_{25}$ is the usual sliver along the longitudinal 25 directions and

$$\mathcal{N}' = \sqrt{3}\frac{V_{00} + \frac{b}{2}}{(2\pi b^3)^{\frac{1}{4}}}\sqrt{\det(1 - X')\det(1 + T')} \tag{7.2}$$

In order to study the space profile of this solution in the transverse direction we contract it with the x_0–coordinate eigenstate

$$|x_0\rangle = \left(\frac{2}{b\pi}\right)^{\frac{1}{4}} \exp\left[-\frac{1}{b}x_0^2 - \frac{2}{\sqrt{b}} i a_0^\dagger x_0 + \frac{1}{2}(a_0^\dagger)^2\right] |\Omega_b\rangle \tag{7.3}$$

The result is

$$\langle x_0|\Lambda'\rangle = \left(\frac{2}{b\pi}\right)^{\frac{1}{4}} \frac{\mathcal{N}'}{\sqrt{1+s'}} \exp\left[\frac{1}{b}\frac{s'-1}{s'+1}x_0^2 - \frac{2i}{\sqrt{b}}\frac{x_0 f_0}{1+s'} - \frac{1}{2}a^\dagger W' a^\dagger\right] \tag{7.4}$$

where the condensed notation means

$$f_0 = \sum_{n=1} S'_{0n} a_n^\dagger, \qquad a^\dagger W' a^\dagger = \sum_{n,m=1} a_n^\dagger W'_{nm} a_m^\dagger, \qquad W'_{nm} = S'_{nm} - \frac{S'_{0n} S'_{0m}}{1+s'} \tag{7.5}$$

and

$$s' = S'_{00} \tag{7.6}$$

After an inverse Wick–rotation $x_0 \to i\mathrm{x}_0, a_n^\dagger \to i a_n^\dagger$ (7.4) becomes

$$\langle \mathrm{x}_0|\Lambda'\rangle = \left(\frac{2}{b\pi}\right)^{\frac{1}{4}} \frac{\mathcal{N}'}{\sqrt{1+s'}} \exp\left[\frac{1}{b}\frac{1-s'}{1+s'}\mathrm{x}_0^2 + \frac{2i}{\sqrt{b}}\frac{\mathrm{x}_0 f_0}{1+s'} + \frac{1}{2}a^\dagger W' a^\dagger\right] \tag{7.7}$$

We are interested in solutions localized in time. The second term in the exponent gives rise to time oscillations. Only the first term can guarantee time localization. Precisely this happens when $|s'| > 1$. However such a condition can never be achieved within the present scheme in which ordinary lump solutions are utilized. In fact it is possible to show that for such solutions $|s'| \leq 1$. Therefore with the simple–minded scheme considered so far it is impossible to achieve time localization (in this regard our negative conclusion is similar to [97]; as for the case $b \to 0$, see below).

Let us see this in more detail by showing that $|s'| \leq 1$. Using the diagonal basis of chapter 3 (see also appendix B) we can write

$$s' \equiv S'_{00} = \int_{-\infty}^{\infty} dk\, V_0^{(k)}(-e^{-\frac{\pi|k|}{2}})V_0^{(k)} + V_0^{(\xi)} e^{-|\eta|} V_0^{(\xi)} + V_0^{(\bar{\xi})} e^{-|\eta|} V_0^{(\bar{\xi})} \tag{7.8}$$

Using (B.7), one can see that the first term in the RHS does not contribute in the limit $b \to 0$ (i.e. $\eta \to 0$) and using the approximants (7.10) we immediately see that the remaining two terms add up to 1. Therefore when $b \to 0$, $s' \to 1$. Viceversa, in the limit $b \to \infty$, using (7.11) we see that the last two terms in the RHS of (7.8) do not contribute, while the first term contribute exactly –1. This can be also shown numerically or with the alternative analytical method of Appendix B. For generic values of b we cannot calculate s' analytically but it is easy to evaluate it numerically and to show that it is a monotonically decreasing function of b for $0 \leq b < \infty$. This in turn implies that the quantity $\frac{1-s'}{1+s'}$ is always *positive* (see figure 1).

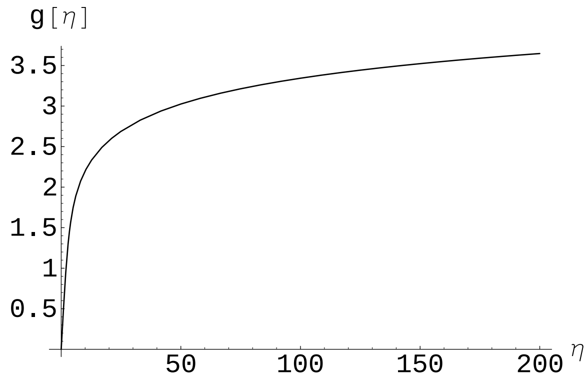

Figure 7.1: *The quantity* $g[\eta] = \frac{1-s'}{1+s'}$ *as a function of* η

Our conclusion is therefore that we cannot obtain a time–localized solution by inverse–Wick–rotating an ordinary lump solution. Some drastic change has to be made in order to produce a localized time–dependent solution.

7.2 Inverse slivers and inverse lumps

Before to discuss the problem of how to find sensible time–localized projectors, we would like to point out that there is another twist invariant solution to the projector equation (4.11), i.e. $1/T$. In fact (4.11) is invariant under the substitution $T \leftrightarrow 1/T$. $1/T$ is given by the RHS of eq.(4.12) with the – sign replaced by the + sign in front of the square root. We will call it the *inverse sliver*. This solution was previously discarded, [54], because of the bad asymptotic behaviour of the $1/T$ eigenvalues. However it is exactly this behaviour that will allow us, in the precise sense clarified later, to find interesting time–dependent solutions[1].

Exactly as in the sliver case, we can consider the solution with T' replaced by $1/T'$. The same considerations hold as in that case.

Using the diagonal basis of the three–strings vertex, discussed in chapter 3, the exponent of the conventional lump state can be written

$$\begin{aligned} a^\dagger S' a^\dagger &= \int_{-\infty}^{\infty} dk\, t(k)\, (a_k^\dagger, C a_k^\dagger) + 2\, t_\xi\, (a_\xi^\dagger, C a_\xi^\dagger) \\ &= \frac{1}{2} \int_{-\infty}^{\infty} dk\, t(k)\, (e_k^\dagger e_k^\dagger + o_k^\dagger o_k^\dagger) + \, t_\eta\, (e_\eta^\dagger e_\eta^\dagger + o_\eta^\dagger o_\eta^\dagger) \end{aligned} \tag{7.9}$$

[1]Notwithstanding the divergent behaviour of the eigenvalues it is perhaps possible to associate a definite meaning to the energy density of some of these solutions. They may be interesting as solutions of VSFT also without reference to time dependence.

where $t(k) = -e^{\pi|k|/2}$ and $t_\eta \equiv t_\xi = e^{-|\eta|}$. As just discussed, we can separately invert continuous and discrete eigenvalues while preserving the projector properties of the state.

In the sequel we need the behaviour of the eigenvectors when $b \to 0$ and when $b \to \infty$. Near $b = 0$ we have

$$\begin{aligned} &b \approx 0, \qquad \eta \approx 0, \qquad \xi \approx 1 \\ &V_0^{(\xi)} = \frac{1}{\sqrt{2}} + \mathcal{O}(\eta), \qquad V_n^{(\xi)} = \mathcal{O}(\eta^2) \end{aligned} \tag{7.10}$$

The same behaviour holds for the $V^{(\bar{\xi})}$ basis.

When $b \to \infty$ we have instead

$$\begin{aligned} &b \to \infty, \qquad b \approx 4\log\eta, \qquad \xi \approx -e^{\frac{\pi i}{3}} \\ &V_0^{(\xi)} \approx e^{-\frac{\eta}{2}}\sqrt{2\eta\log\eta}, \qquad V_n^{(\xi)} \sim e^{-\frac{\eta}{2}}\sqrt{\eta} \end{aligned} \tag{7.11}$$

and the same for $V^{(\bar{\xi})}$.

These asymptotic behaviors will be used to evaluate matrix elements such as (7.8). In this regard they are completely reliable (and, in any case, backed up by numerical evidence). If we consider instead the corresponding asymptotic expansions for the $V^{(k)}$ basis, we have to be more careful. The point is that the expression $(V_0^{(k)})^2$, see (B.7), would superficially seem to vanish in the limit $b \to \infty$, but it is in fact a representation of the Dirac delta function $\delta(k)$, see Appendix B. Therefore the result of taking the $b \to \infty$ limit in an integral containing $(V_0^{(k)})^2$ is to concentrate it at the point $k = 0$. This renders the generating function (B.6) very singular and, consequently, such integrals as $\int dk\, V_n^{(k)} V_m^{(k)} f(k)$ must be handled with care. As for the limit of the continuous basis when $b \to 0$, one can see that $V_0^{(k)} \to 0$, while the other eigenfunctions have a nonvanishing finite limit.

7.3 A Rolling Tachyon–like Solution

It is not hard to realize that if we were to replace $e^{-|\eta|}$ with $e^{|\eta|}$ in eq.(7.8) we would reverse the conclusion at the end of the previous section. In fact, see below, we would have $|s'| \geq 1$. In this section we wish to exploit this possibility. In section 2 we have seen that if in the lump solution we replace T' by $1/T'$, formally, we still have a projector. Motivated by this fact we define an unconventional lump, by replacing $|\check{\Lambda}'\rangle$ in (7.1) with

$$|\check{\Lambda}'\rangle = \check{\mathcal{N}}'\exp\left(-\frac{1}{2}a^\dagger C\check{T}'a^\dagger\right)|\Omega_b\rangle \tag{7.12}$$

where

$$\check{T}'_{NM} = -\int_{-\infty}^{\infty} dk\, V_N^{(k)}\, V_M^{(k)} \exp\left(-\frac{\pi|k|}{2}\right) + \left(V_N^{(\xi)}\, V_M^{(\xi)} + V_N^{(\bar{\xi})}\, V_M^{(\bar{\xi})}\right)\exp|\eta| \tag{7.13}$$

Due to the fact that the star product is split into eigenspaces of the Neumann coefficients X', X'_+, X'_-, the projector equation split accordingly into the continuous and discrete spectrum part. Therefore we are guaranteed that (7.12) is again a projector, as one can on the other hand easily verify by direct calculation. This is the solution we propose.

Before we proceed with our analysis we would like to clarify a basic question about the solution we have just put forward. Passing from a squeezed state solution with a matrix T' to another characterized by the inverse matrix $1/T'$ may lead in general to unacceptable features of the state, such as divergent terms in the oscillator basis. However in the case at hand, in which one inverts only the discrete spectrum, such unpleasant aspects disappear. First of all the matrix $\check{T}'$ is well defined both in the oscillator and in the diagonal basis. Second, such expression as $\sqrt{\det(1-\check{T}')}$ are well–defined. This is due to the fact that, if we are allowed to factorize the discrete and continuous spectrum contribution, the former can be written as $\det(1-\check{T}')^{(d)} = (1-\exp|\eta|)^2$, so that the possible dangerous – sign under the square root disappears due to the double multiplicity of the discrete eigenvalue. Third, the energy density of the (Euclidean) solution (7.12) equals the energy density of the ordinary lump. In fact, using the formulas of [54], the ratio between the energy densities of the two solutions reduces to

$$\sqrt{\frac{\det(1+\check{T}')}{\det(1-\check{T}')}}\Big/\sqrt{\frac{\det(1+T')}{\det(1-T')}} = \sqrt{\frac{(1+e^{|\eta|})^2}{(1-e^{|\eta|})^2}}\Big/\sqrt{\frac{(1+e^{-|\eta|})^2}{(1-e^{-|\eta|})^2}} = 1 \tag{7.14}$$

after factorization of the discrete and continuous parts of the spectrum.

After these important remarks it remains for us to show that this solution has the appropriate features to represent a rolling tachyon solution. To see if this is true we have to represent it in a more explicit way. In particular we have to extract the explicit time dependence (better, the space dependence and then inverse–Wick–rotate it). To do so, we have to choose a (coordinate) basis on which to project (7.12). There seem to be two distinguished ways to make this choice. We will work them out explicitly and then discuss them.

To start with let us define the following coordinate and momentum operator, given by the twist even and twist odd parts of the discrete spectrum,

$$\hat{x}_\eta = \frac{i}{\sqrt{2}}(e_\eta - e_\eta^\dagger) \tag{7.15}$$

$$\hat{y}_\eta = \frac{i}{\sqrt{2}}(o_\eta - o_\eta^\dagger) \tag{7.16}$$

The eigenstates of the coordinate $\hat{x}_\eta$ are given by

$$|x\rangle = \frac{1}{\sqrt{\pi}}\exp\left(-\frac{1}{2}x^2 - \sqrt{2}ie_\eta^\dagger x + \frac{1}{2}e_\eta^\dagger e_\eta^\dagger\right)|\Omega_{\eta_e}\rangle, \tag{7.17}$$

$$e_\eta|\Omega_{\eta_e}\rangle = 0$$

$$\hat{x}_\eta|x\rangle = x|x\rangle$$

Correspondingly the eigenstates of the momentum $\hat{y}_\eta$ are

$$|y\rangle = \frac{1}{\sqrt{\pi}}\exp\left(-\tfrac{1}{2}y^2 - \sqrt{2}io_\eta^\dagger y + \tfrac{1}{2}o_\eta^\dagger o_\eta^\dagger\right)|\Omega_{\eta_o}\rangle, \tag{7.18}$$

$$o_\eta|\Omega_{\eta_o}\rangle = 0$$

$$\hat{y}_\eta|y\rangle = y|y\rangle$$

In order to make the x, y dependence explicit we project our solution (7.12) into the position/momentum eigenstates (7.17, 7.18). Using standard results[2] we get

$$\langle x, y|\check{\Lambda}'\rangle = \frac{1}{\pi(1+e^{|\eta|})}\exp\left(\frac{e^{|\eta|}-1}{e^{|\eta|}+1}(x^2+y^2)\right)|\check{\Lambda}'_c\rangle \tag{7.19}$$

The state $|\check{\Lambda}'_c\rangle$ is given by (7.12), but with only oscillators from the continuous spectrum, as the contribution of the discrete spectrum is now contained in the prefactor at the rhs of (7.19) . Now we perform the inverse Wick rotation $x \to i\mathrm{x}$, $y \to -i\mathrm{y}$ to recover the Lorentz signature, and obtain

$$|\check{\Lambda}'(\mathrm{x};\mathrm{y})\rangle = \frac{1}{\pi(1+e^{|\eta|})}\exp\left(-\frac{e^{|\eta|}-1}{e^{|\eta|}+1}(\mathrm{x}^2+\mathrm{y}^2)\right)|\check{\Lambda}'_c\rangle^{(Wick)} \tag{7.20}$$

It is evident that for every value of η the solution is localized in the x–time coordinate. The extra coordinate y is related to internal twist odd degrees of freedom and can be interpreted as a free parameter of the representation (7.20). This solution also contains the free parameter η which is nothing but a reparametrization of b, through (B.3). Therefore it is characterized by two free parameters.

The 'time' x is not the ordinary time, i.e. the time coupled to the flat open string metric and related to the string center of mass. We will see later on a possible interpretation for x. Now, let us turn to the ordinary (open string) time, i.e. the time defined by the center of mass of the string and analyze the corresponding time profile. Despite the fact that this coordinate is not diagonal for the $*$–product we can still have complete control on the profile along it. The center of mass position operator is given by

$$\hat{x}_0 = \frac{i}{\sqrt{b}}(a_0 - a_0^\dagger) \tag{7.21}$$

The center of mass position eigenstate is

$$|x_0\rangle = \left(\frac{2}{b\pi}\right)^{\frac{1}{4}}\exp\left(-\frac{1}{b}x_0x_0 - \frac{2}{\sqrt{b}}ia_0^\dagger x_0 + \frac{1}{2}a_0^\dagger a_0^\dagger\right)|\Omega_b\rangle \tag{7.22}$$

Let us compute the center of mass time profile. After inverse–Wick–rotating it, it turns out to be

$$|\check{\Lambda}'(\mathrm{x}_0)\rangle \;=\; \langle \mathrm{x}_0|\check{\Lambda}'\rangle = \tag{7.23}$$

[2] Here we are assuming that the vacuum factorizes into $|\Omega_{\eta_e}\rangle \otimes |\Omega_{\eta_o}\rangle \otimes |\Omega_c\rangle$ where the latter factor represents the vacuum with respect to the continuous oscillator component.

$$\left(\frac{2}{b\pi}\right)^{\frac{1}{4}} \frac{\check{\mathcal{N}}'}{\sqrt{1+\check{T}'_{00}}} \qquad \exp\left(\frac{1}{b}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}\mathrm{x}_0^2 + \frac{2i}{\sqrt{b}(1+\check{T}'_{00})}\mathrm{x}_0\check{T}'_{0n}a_n^\dagger + \frac{1}{2}a_n^\dagger W'_{nm}a_m^\dagger\right)|\Omega_b\rangle$$

$$W'_{nm} = \check{S}'_{nm} - \frac{1}{1+\check{T}'_{00}}\check{S}'_{0n}\check{S}'_{0m} \tag{7.24}$$

The quantities $\check{S}'_{0n}$ and $\check{S}'_{nm}$ can be computed in the diagonal basis

$$\begin{aligned}\check{S}'_{0n} &= \check{T}'_{0n} &(7.25)\\ &= (1+(-1)^n)\left(-\int_0^\infty V_0^{(k)}V_n^{(k)}\exp\left(-\frac{\pi k}{2}\right) + V_0^{(\xi)}V_n^{(\xi)}\exp|\eta|\right)\\ \check{S}'_{nm} &= (-1)^n\check{T}'_{nm} = &(7.26)\\ &= ((-1)^n+(-1)^m)\left(-\int_0^\infty V_n^{(k)}V_m^{(k)}\exp\left(-\frac{\pi k}{2}\right) + V_n^{(\xi)}V_m^{(\xi)}\exp|\eta|\right)\end{aligned}$$

It is evident that the leading time dependence in (7.23), for large x_0, is contained in $\exp\left(\frac{1}{b}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}\mathrm{x}_0^2\right)$. The number $\check{T}'_{00}$ is $b(\eta)$–dependent and can be computed via

$$\check{T}'_{00}(\eta) = -2\int_0^\infty dk\left(V_0^{(k)}(b(\eta))\right)^2\exp\left(-\frac{\pi k}{2}\right) + 2(V_0^{(\xi)})^2\exp|\eta| \tag{7.27}$$

This is the crucial quantity as far as the time profile is concerned. An analytic evaluation of it is beyond our reach. However we will later show that

$$\lim_{\eta\to 0}\check{T}'_{00} = 1 \tag{7.28}$$

$$\lim_{\eta\to\infty}\check{T}'_{00} = \infty \tag{7.29}$$

A numerical analysis shows that this quantity is a function monotonically increasing with η within such limits. This means that the quantity $\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}$ is always *negative* (it lies in the interval $(-1,0)$, see figure 2) and so the profile is always localized in the center of mass time, except in the extreme case $\eta\to 0$, which corresponds to the tensionless limit.

This has to be compared with the usual lump solution (see previous section) for which the corresponding quantity is always positive and takes values in the interval $(0,\infty)$, allowing for localized *space* profiles but divergent along a timelike direction.

For reasons that will become clear in the next section, we extract also the free parameter y dependence, by projecting onto the corresponding twist–odd eigenstate (7.18). This operation can be done before or after the projection along the center of mass coordinate and does not interfere with it because $\hat{\mathrm{y}}$ does not contain the zero mode. We will therefore consider the following representation of our solution (inverse Wick rotation is again understood)

$$|\Lambda'(\mathrm{x}_0,y)\rangle = \langle\mathrm{x}_0,\mathrm{y}|\check{\Lambda}'\rangle = \left(\frac{2}{b\pi}\right)^{\frac{1}{4}}\frac{\check{\mathcal{N}}'}{\sqrt{2\pi(1+e^{|\eta|})}}\exp\left(\frac{1-e^{|\eta|}}{1+e^{|\eta|}}\mathrm{y}^2\right) \tag{7.30}$$

$$\cdot\frac{1}{\sqrt{1+\check{T}'_{00}}} \qquad \exp\left(\frac{1}{b}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}\mathrm{x}_0^2 + \frac{2i}{\sqrt{b}(1+\check{T}'_{00})}\mathrm{x}_0\check{T}'_{0n}a_n^\dagger - \frac{1}{2}a_n^\dagger W''_{nm}a_m^\dagger\right)|0\rangle$$

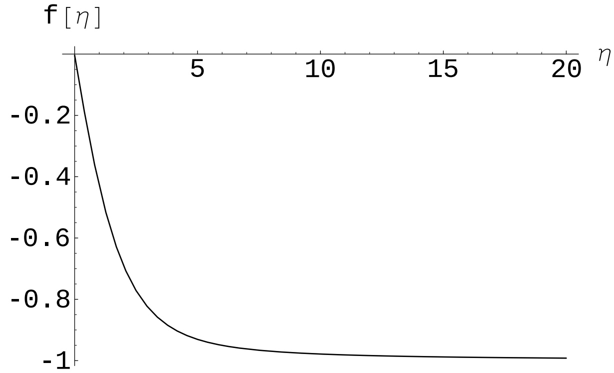

Figure 7.2: *The quantity* $f[\eta] = \frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}$ *as a function of* η

The quantities $\check{T}'_{00}$ and $\check{T}'_{0n}$ are the same as in (7.27, 7.25) since the momentum $\hat{y}_\eta$ is twist–odd. Some changes occur in W''_{nm}

$$
\begin{aligned}
W''_{nm} &= \check{S}''_{nm} - \frac{1}{1+\check{T}'_{00}}\check{S}'_{0n}\check{S}'_{0m} && (7.31)\\
\check{S}''_{nm} &= ((-1)^n + (-1)^m)\left(-\int_0^\infty dk\, V_n^{(k)}\, V_m^{(k)} \exp\left(-\frac{\pi k}{2}\right) + V_n^{(\eta)} V_m^{(\eta)} \exp|\eta|\right) \quad n, m \text{ even}\\
&= -((-1)^n + (-1)^m)\int_0^\infty dk\, V_n^{(k)}\, V_m^{(k)} \exp\left(-\frac{\pi k}{2}\right) \quad n, m \text{ odd}
\end{aligned}
$$

Note that $\check{S}''_{nm}$ gets contribution only from the twist–even part of the discrete spectrum.

In conclusion (7.30) provides the solution we were looking for. It represents a solution localized in x_0, with the desired profile. It depends on two free parameters y and η (or b). These are all positive features. But let us start making a closer comparison with the rolling tachyon solution (such a comparison is made with the representation (7.30)). This can be done by considering the limit $b \to \infty$, which can be derived from the eqs.(7.11). $b \to \infty$ means $\eta \to \infty$ (for simplicity from now on we take η positive) and

$$
\check{T}'_{00} \approx 2\,\eta \log \eta \,(1 - \frac{\log(2\pi)}{\log \eta} + \ldots) \tag{7.32}
$$

where dots denote higher order terms. Therefore we see that in this limit any time dependence in (7.30) disappears. Moreover, anticipating a result of the following section, we also have that $W''_{nm} \to S_{nm}$. In other words, in the limit $b \to \infty$ we obtain a static solution corresponding to the initial sliver. From this we understand that the parameter $1/b$, for large b, plays a role similar to Sen's parameter $\tilde{\lambda}$ near 0[3]. A second remark concerns the limit $y \to \infty$. In this case the first exponential factor in the RHS of (7.30) suppresses

[3] We recall that Sen's rolling tachyon solution depends on the parameter $\tilde{\lambda}$, which appears in the $\tilde{\lambda} \int_{\partial D} dt \cosh X^0(t)$–deformed BCFT.

everything, so that the limit is the 0 state. In other words, we can consider this value of the parameter y as identifying the (relatively) stable vacuum state.

Although the rolling tachyon naturally compares with (7.30) rather than with (7.20), it is instructive to repeat something similar with the latter. Let us stress once more that both (7.30) and (7.20) represent the same solution, but in different bases, in particular with two different times: one, x_0, is the open string center of mass time, the other, x. is related to the discrete spectrum. In the (7.20) case a parameter like b is missing. But this is something that is simply not customary and can be easily remedied. We can in fact introduce a parameter b_e in (7.15,7.16), just replacing $\sqrt{2}$ with $\sqrt{b_e}$ in those equations. Then (7.20) would become

$$|\check{\Lambda}'(\mathrm{x};\mathrm{y})\rangle = \frac{1}{b_e\pi(1+e^\eta)}\exp\left(-\frac{1}{b_e}\frac{e^\eta-1}{e^\eta+1}(\mathrm{x}^2+\mathrm{y}^2)\right)|\check{\Lambda}'_c\rangle^{(Wick)} \tag{7.33}$$

and we could repeat the same argument as above and reach the same conclusion, except that in this case we have to take $b_e \to \infty$ as well as $\eta \to \infty$. The limit $y \to \infty$ plays the same role as in the (7.30) representation.

In the next section we will study the solution (7.12) in a regime we are more familiar with, the low energy regime $\alpha' \to 0$, and in the other extreme regime, $\alpha' \to \infty$, in which the solution considerably simplifies and an analytic treatment is possible. What we would like to see more closely is whether, for sufficiently small values of the parameters, the solution at time 0 is close enough to the sliver configuration (4.8), whose decay the solution is expected to describe.

7.4 Low energy and tensionless limits

As reviewed in appendix B, the low energy limit is obtained by performing an $\epsilon \to 0$ limit on the quantities that depend on the Neumann coefficients of the three strings vertex. ϵ is a dimensionless parameter that represents the smallness of α', [64]. As it happens, in all the expansions we consider, the parameters ϵ and b only appear through the ratio ϵ/b. Therefore, formally, the expansions for small ϵ/b are the same as the expansions for large b, i.e. $\eta \to \infty$. Therefore, in this section, when we consider the expansion in η near ∞ we really mean the expansion for ϵ/b small (i.e. ϵ small and b finite). A different attitude is required by the 'external' states like (7.3). There the rescaling of x_0 would lead to the replacement $b \to b\epsilon$. In this case we absorb ϵ into x_0 and keep b finite. In conclusion, throughout the analysis of the low energy limit, b should be considered as a finite free parameter.

Let us analyze in detail what is the limit of the various quantities appearing in (7.30). First of all we have

$$\lim_{\eta\to\infty}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}} = -1 \tag{7.34}$$

This follows from (7.11) and from the discussion at the end of section 3, in particular from the property of $(V_0^{(k)})^2$ of approximating $\delta(k)$ in the limit $b \to \infty$, which implies that $\check{T}'_{00} \to \infty$ in the same limit. For the oscillating term we have

$$\lim_{\eta\to\infty} \frac{\check{T}'_{0n}}{1+\check{T}'_{00}} = \lim_{\eta\to\infty} \frac{1}{\sqrt{2\log\eta}} = 0 \tag{7.35}$$

To evaluate this limit one must evaluate $\check{T}'_{0n}$. This in turn requires knowing the asymptotic expansion of the basis $V_n^{(k)}$ for $\eta \to \infty$. This is done in Appendix B. A numerical approximation confirms the above result.

Thus, in the limit, the oscillating part completely decouples from the time dependent part. It remains for us to consider the limit of the quadratic form W''_{nm}, (7.31). When n, m are odd there are no contributions from the discrete spectrum, since the contraction with $\langle \mathrm{y}|$ has eliminated them.

$$W''_{2n-1,2m-1} = \check{S}'^{(c)}_{2n-1,2m-1}, \tag{7.36}$$

When n, m are even we have, on the contrary, potentially dangerous terms because there are divergent contributions arising from the discrete spectrum. The latter have to be carefully evaluated.

$$\begin{aligned} W''^{(d)}_{2n,2m} &= 2V^{(\xi)}_{2n}V^{(\xi)}_{2m}\left(e^{\eta} - \frac{2(V_0^{(\xi)})^2 e^{2\eta}}{2(V_0^{(\xi)})^2 e^{\eta} + \mathcal{O}(\frac{e^{-\eta}}{\eta\log\eta})}\right) \\ &= 2V^{(\xi)}_{2n}V^{(\xi)}_{2m}\,\mathcal{O}(\frac{e^{-\eta}}{\eta\log\eta}) \approx \mathcal{O}(\frac{1}{\log\eta}) \end{aligned} \tag{7.37}$$

We see that the potentially divergent contributions arising from the discrete spectrum exactly cancel when $\eta \to \infty$. Therefore, as far as $W''_{2n,2m}$ is concerned, we are left only with the contribution from the continuous spectrum. Of the two pieces that contribute to $W''^{(c)}_{2n,2m}$, see eq.(7.31) only the first survives in the limit $\eta \to \infty$, the second vanishes for the usual reasons. Therefore we can conclude that

$$W''_{nm} = \check{S}'^{(c)}_{nm} + \ldots$$

where dots denote subleading corrections of order at least $1/\log\eta$. At this stage we can do the calculation directly as in Appendix B, or we can resort to an indirect argument by noticing that $\check{S}'^{(c)}_{nm}$ approaches S'_{nm} in the same limit, because the discrete spectrum contribution to the latter vanishes, and then use the results of Appendix B. In both cases we conclude that

$$W''_{nm} = S_{nm} + \mathcal{O}(\epsilon/b) \tag{7.38}$$

Going back to equation (7.30) we see that, modulo a normalization factor, we obtain

$$\lim_{\alpha'\to 0} |\check{\Lambda}'(\mathrm{x}_0,\mathrm{y})\rangle = \check{\mathcal{N}}'(\mathrm{y})\, e^{-\frac{\mathrm{x}_0^2}{b}}\, |\Xi\rangle \tag{7.39}$$

where $|\Xi\rangle$ is the zero momentum sliver state. This result can be phrased as follows: in the low energy limit the solution takes the form of a time–Gaussian multiplying a sliver, the subleading terms being proportional to ϵ/b, eq.(7.38).

To end this section let us briefly consider the opposite limit, that is $\alpha' \to \infty$ (tensionless limit). As in the previous case this is formally achieved by taking the $\eta \to 0$ limit in all the quantities which are related to the Neumann coefficients, but leaving b as a free parameter. This limit is well defined. Using the results of appendix C we get

$$\lim_{\eta \to 0} \frac{1 - \check{T}'_{00}}{1 + \check{T}'_{00}} = 0 \tag{7.40}$$

The oscillating term in (7.30) vanishes as well. This result implies that the Gaussian representing time dependence in (7.30) is actually completely flat: time dependence has disappeared! We believe this to be related to the fact that all strings modes become massless in this limit [98], so there are no modes to decay into. It is easy to see that the only non vanishing term in the exponent of (7.30) is the quadratic part which gets contribution only from the continuous spectrum (on the contrary of the $\eta \to \infty$ limit the discrete eigenvector has only the 0–component, while the higher components disappear like positive powers of $1/\eta$). We remark that in the tensionless limit the center of mass time and the x time are identified.

7.5 Discussion

In the last two sections we have shown that by inverting the discrete part of the spectrum we obtain a definite (unconventional) lump solution which, after inverse Wick–rotation, gives rise to a time–localized state with many properties characteristic of the rolling tachyon solution. In the course of our exposition we have left aside some loose ends which we would like now to tie up or at least comment upon.

The first comment concerns normalization of the states we have come across. We have written down throughout normalization factors in quite a formal way. We have already recalled the fact that the sliver state and the lump state have a vanishing normalization, but we believe these problems have to be kept separated from the normalization of our time dependent solution. As a matter of fact a normalization problem appears only for the representation (7.30) and in the low energy limit, for the coefficient $\check{\mathcal{N}}'$ in (7.39) diverges exponentially for $\eta \to \infty$ once all the contributions are taken into account (this problem does not arise for the other representation (7.20)). We remark however that, as was noticed in the discussion after eq.(7.12), the energy density of the corresponding Euclidean solution is well–defined (once the conventional lump energy density is). Therefore the exploding normalization can only be an artifact of the representation. It means that we have to use the parameters of the state to regulate the normalization, although it is not clear a priori what is the right way to do it. A possibility is to use the factor $\exp\left(\frac{1-e^{|\eta|}}{1+e^{|\eta|}}\mathrm{y}^2\right)$ in (7.30). Since this vanishes for y large, we can view y as a suitable function of η as $\eta \to \infty$. This can settle the problem. Other possibilities are connected to dressing, [3, 4].

We would like to add a comment concerning the meaning of our solution (7.12) before inverse Wick–rotation. As we have noticed, its profile is an inverted Gaussian that explodes at infinity. This suggests that we can interpret it as a D–brane located at infinity in the transverse direction, that is at infinite imaginary time. One could speculate this to be linked to the D–branes at imaginary times referred to in [76, 91].

Another important question is the number of parameters. Our solution depends on two parameters y and b. One may wonder why we extracted the y dependence from 7.12. This is indeed not a choice but a constraint. Had we not done it, we would have found a different formula (7.31) in which also the n, m odd part of S''_{nm} would have taken a contribution from the discrete spectrum (exactly as the n, m even part). However in the odd–odd part no such cancelation (7.37) as in the even–even part occurs and we would find badly divergent coefficients in W'. We gather that y is a genuine free parameter of the time–dependent state. What about b? It was argued in [54] that this parameter represents a gauge degree of freedom. This need not be in contradiction with the meaning we have attributed to it in the previous sections. We recall that in ordinary gauge theory a singular gauge transformation may convey some physical information. Now, looking at (2.44), the values $b = 0$ and $b = \infty$ may well correspond to singular gauge transformations, and therefore contain physical information. More generally the gauge nature of b may mean that using a different formulation one may be able to write the solution in terms of a single physical parameter which contains the information carried by both b and y.

The third question we would like to address is the relation between the two representations (7.20) and (7.30). The latter is expressed in terms of the open string center of mass x_0 and its interpretation is obvious. The interpretation of the former is less clear since the 'time' x does not have a clear connection with the open string center of mass time. A rather bold speculation is that x be connected with the closed string time. The closed string time couples to the closed string metric, which, in correspondence with the D–brane, must develop a singularity (it must be a solution of the effective low energy field theory associated to the closed string). So the relation between the open and the closed string time should be something like $g_c(dt_c)^2 \sim g_o(dt_o)^2$ in the field theory limit, were $g_o = 1$ and g_c becomes larger and larger near the origin. Something similar indeed occurs between x_0 and x when $\eta \to \infty$. In fact the ratio between x_0 and the zero mode part of x decreases exponentially with η. We notice moreover that the normalization of the representation (7.20) does not need any regularization. In other words x seems to be a smoother choice of time, with respect x_0.

Next we would like to recall that recently, [95], the role of the time coordinate represented by the midpoint $X^0\left(\frac{\pi}{2}\right)$ for causality in SFT has been emphasized. In our VSFT solution the profile along this time turns out to be highly singular: it is a constant infinite function (finite only at $X^0\left(\frac{\pi}{2}\right) = 0$), the inverse Wick–rotation of the midpoint space profile of [64].

7.6 Adding a longitudinal E–field

In this section we will analyze the case of switching the $E-$field along a tangential direction, i.e., along, say, the world volume of a $D25-$brane. As explained in [36], the presence of the E–field does not create non commutativity as the direction in which it is turned on is at zero momentum.

We use the double Wick rotation, that is we make space–time euclidian by sending $X^0(\sigma) \to iX^D(\sigma)$; then we construct an *unconventional* lump solution, [5], on the transverse spatial direction $X^D(\sigma)$ and inverse Wick rotate along it, $X^D(\sigma) \to -iX^0(\sigma)$. Let $\alpha, \beta = 1, D$ be the couple of directions on which the E–field is turned on. Then E–field physics is obtained by taking an imaginary B–field

$$B_{\alpha\beta} = B\epsilon_{\alpha\beta} = iE\epsilon_{\alpha\beta}, \quad E \in \Re \tag{7.41}$$

A localized time dependent solution is easily given by straightforwardly changing the metric $\eta_{\alpha\beta}$ of the solution of [5], with the open string metric $G_{\alpha\beta}$

$$G_{\alpha\beta} = (1 - (2\pi\alpha' E)^2)\, \delta_{\alpha\beta} \tag{7.42}$$

$$G^{\alpha\beta} = \frac{1}{1 - (2\pi\alpha' E)^2}\, \delta^{\alpha\beta} \tag{7.43}$$

Note that, contrary to the case of a real B–field, a critical value shows up for the imaginary analytic continuation[4]

$$E_c = \frac{1}{2\pi\alpha'} \tag{7.44}$$

From now on all indexes (α, β) are raised/lowered with the open string metric (7.126).

We have then the following commutators

$$[a^{\alpha}_m, a^{\beta\dagger}_n] = G^{\alpha\beta}\delta_{mn}, \qquad m, n \geq 1 \tag{7.45}$$

stating that the a^{α}'s are canonically normalized with respect the open string metric (7.126)

We recall that, in case of a background $B_{\alpha\beta}$–field, the three string vertex is deformed to be, [42] (see also [99])

$$|V_3\rangle = |V_{3,\perp}\rangle \otimes |V_{3,_{\|}}\rangle \tag{7.46}$$

[4] In the rest of the paper we will set $\alpha' = 1$

The factor $|V_{3,\|}\rangle$ concerns the directions with no B–field and its expression is the usual one, [52, 14, 71, 1], on the other hand $|V_{3,\perp}\rangle$ deals with the directions on which the B field is turned on[5] .

$$|V_{3,\perp}\rangle = \int d^{26}p_{(1)}d^{26}p_{(2)}d^{26}p_{(3)}\delta^{26}(p_{(1)}+p_{(2)}+p_{(3)})\exp(-E')\,|0,p\rangle_{123} \tag{7.47}$$

The operator in the exponent is given by, [42]

$$\begin{aligned} E'_{\perp} &= \sum_{r,s=1}^{3}\left(\frac{1}{2}\sum_{m,n\geq 1}G_{\alpha\beta}a_m^{(r)\alpha\dagger}V_{mn}^{rs}a_n^{(s)\beta\dagger}+\sum_{n\geq 1}G_{\alpha\beta}p_{(r)}^{\alpha}V_{0n}^{rs}a_n^{(s)\beta\dagger}\right. \\ &\qquad \left.+\frac{1}{2}G_{\alpha\beta}p_{(r)}^{\alpha}V_{00}^{rs}p_{(s)}^{\beta}+\frac{i}{2}\sum_{r<s}p_{\alpha}^{(r)}\theta^{\alpha\beta}p_{\beta}^{(s)}\right) \end{aligned} \tag{7.48}$$

Note that the part giving rise to space–time non–commutativity, $\frac{i}{2}\sum_{r<s}p_{\alpha}^{(r)}\theta^{\alpha\beta}p_{\beta}^{(s)}$, does not contribute due to the zero momentum condition in the 1 spatial direction.
Let's first consider the sliver solution at zero momentum along the 1 direction
The three string vertex in such a direction takes the form ($p^1=p_1=0$)

$$|V_3(E,p=0)\rangle = |V_3(E=0,p=0)\rangle^{(\eta_{11}\to G(E)_{11})} \tag{7.49}$$

$$= \exp\left(\frac{1}{2}\sum_{r,s=1}^{3}G_{11}a^{(r)1\dagger}\cdot V^{rs}\cdot a^{(s)1\dagger}\right)|0\rangle \tag{7.50}$$

This implies that the zero momentum sliver is in this case

$$|S(E,p=0)\rangle = |S(E=0,p=0)\rangle^{(\eta_{11}\to G(E)_{11})} \tag{7.51}$$

$$= \mathcal{N}\exp\left(-\frac{1}{2}G_{11}a^{1\dagger}\cdot S\cdot a^{1\dagger}\right)|0\rangle \tag{7.52}$$

where the normalization $\mathcal{N}$ and the matrix S are given as usual, [54],

$$T = CS = \frac{1}{2X}(1+X-\sqrt{(1+3X)(1-X)}) \tag{7.53}$$

$$\mathcal{N} = \sqrt{\det(1-X)(1+T)} \tag{7.54}$$

On the euclidian time direction we need the full 3 string vertex in oscillator basis. This is given by

$$|V_{3,\perp}\rangle' = K\,e^{-E'}|\Omega_b\rangle \tag{7.55}$$

with

$$K = \left(\frac{\sqrt{2\pi b^3}}{3(V_{00}+b/2)^2}(1-(2\pi E)^2)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \tag{7.56}$$

$$E' = \frac{1}{2}\sum_{r,s=1}^{3}\sum_{M,N\geq 0}a_M^{(r)D\dagger}V'^{rs}_{MN}a_N^{(s)D\dagger}G_{DD} \tag{7.57}$$

[5]Note that in the case under consideration the symbols $\perp$ and $\|$ do not refer to perpendicular or transverse directions to the brane, but simply indicates directions with E–field turned on ($\perp$) or not ($\|$)

where M, N denote the couple of indices $\{0, m\}$ and $\{0, n\}$, respectively, and D is the (euclidian) time direction. The coefficients V'^{rs}_{MN} are given in Appendix B of [54]. In order to have localization in Minkowski time, we need an explosive profile in euclidian time (unconventional lump); this is explained in detail in the previous section

$$|\check{\Lambda}'\rangle = \mathcal{N}\exp\left(-\frac{1}{2}G_{DD}a^{\dagger D}C\check{T}'a^{\dagger D}\right)|\Omega_b\rangle \tag{7.58}$$

where

$$\check{T}'_{NM} = -\int_{-\infty}^{\infty} dk\, V_N^{(k)}\, V_M^{(k)}\exp\left(-\frac{\pi|k|}{2}\right) + \left(V_N^{(\xi)}\, V_M^{(\xi)} + V_N^{(\bar{\xi})}\, V_M^{(\bar{\xi})}\right)\exp|\eta| \tag{7.59}$$

We refer to chapter 1 and appendix B for the exact definition of eigenvalues and eigenvectors of the various Neumann matrices in the game. We only stress that the Neumann matrix of the unconventional lump has inverted discrete eigenvalues with respect to the ordinary lump: this, as shown before is what guarantees time localization with respect to the center mass and to the time coordinates identified by the discrete eigenvectors $V_N^{(\bar{\xi})}$, $V_N^{\xi)}$.

We get a localized time profile by projecting on the coordinates/momenta of the discrete spectrum

$$\hat{x}_\eta = \frac{i}{\sqrt{2}}\left(e_\eta - e_\eta^\dagger\right) \tag{7.60}$$

$$\hat{y}_\eta = \frac{i}{\sqrt{2}}\left(o_\eta - o_\eta^\dagger\right) \tag{7.61}$$

where e_η / o_η are oscillators constructed with the twist even/odd part of the discrete spectrum eigenvectors $V_N^{(\bar{\xi})}$, $V_N^{(\xi)}$

$$e_\eta = \sum_{N=0}^{\infty}\frac{1}{2}\left(1+(-1)^N\right)V_N^{(\xi)}a_N \tag{7.62}$$

$$o_\eta = \sum_{N=0}^{\infty}\frac{1}{2i}\left(1-(-1)^N\right)V_N^{(\xi)}a_N \tag{7.63}$$

The profile along these coordinates is given by (inverse Wick rotation, $(x, y) \to i(\mathrm{x}, -\mathrm{y})$ is assumed)

$$|\check{\Lambda}'(\mathrm{x},\mathrm{y})\rangle = \langle \mathrm{x},\mathrm{y}|\check{\Lambda}'\rangle = \frac{1}{\pi(1+e^{|\eta|})}\exp\left(-\frac{e^{|\eta|}-1}{e^{|\eta|}+1}(\mathrm{x}^2+\mathrm{y}^2)\right)|\check{\Lambda}'_c\rangle \tag{7.64}$$

where $|\check{\Lambda}'_c\rangle$ contains only continuous spectrum contributions. This profile is localized on the time coordinate x. Note however that there is no more reference to the E–field in the exponent. In order to see explicitly the presence of the E–field, we need to use the usual *open* string time, i.e. the center of mass.

Therefore we contract our solution with the center of mass euclidian time, x^D, and then inverse Wick rotate it, $x^D \to ix^0$. This is identical to to the $E = 0$ case, so we just quote the result, paying attention to use the open string metric (7.126)

$$|\Lambda'(\mathrm{x}_0, y)\rangle \;=\; \langle \mathrm{x}_0, \mathrm{y}|\Xi_\eta\rangle = \sqrt{\frac{2}{b\pi}}\frac{\mathcal{N}}{\sqrt{2\pi(1+e^{|\eta|})}}\exp\left(\frac{1-e^{|\eta|}}{1+e^{|\eta|}}\mathrm{y}^2\right) \tag{7.65}$$

$$\cdot\frac{1}{\sqrt{1+\check{T}'_{00}}} \quad \exp\left(-\mathcal{A}(x^0)^2 + \frac{2i\sqrt{1-(2\pi E)^2}}{\sqrt{b}(1+\check{T}'_{00})}x^0\check{T}'_{0n}\tilde{a}_n^\dagger - \frac{1}{2}\tilde{a}_n^\dagger W''_{nm}\tilde{a}_m^\dagger\right)|0\rangle$$

The extra coordinate y is given by the twist odd contribution of the discrete spectrum, we need to project along it in order to have a well defined $b \to \infty$ limit in the oscillator part W''_{nm}. The oscillators $\tilde{a}_n$ are canonically normalized with respect the η-metric and are given by

$$\tilde{a}_n = \sqrt{1-(2\pi E)^2}a_n \tag{7.66}$$

The quantity that give rise to time localization is then

$$\mathcal{A} = -\frac{1}{b}\frac{1-\check{T}'_{00}}{1+\check{T}'_{00}}(1-(2\pi E)^2) \tag{7.67}$$

This quantity depends on the free parameter b, as well as on the value of the E–field, through the open string metric, used to covariantize the quadratic form in time. The matrix element $\check{T}'_{00}$ is given in [5]

$$\check{T}'_{00}(\eta) = -2\int_0^\infty dk\left(V_0^{(k)}(b(\eta))\right)^2\exp\left(-\frac{\pi k}{2}\right) + 2(V_0^{(\xi)})^2\exp|\eta|, \tag{7.68}$$

it is a monotonic increasing function of b, greater than 1: this is what ensures localization in time as opposed to the standard lump which is suited for space localization.

The life time of the brane is thus given by

$$\Delta T = \frac{1}{2}\sqrt{\frac{1}{2\mathcal{A}}} = \frac{1}{(1-(2\pi E)^2)^{\frac{1}{2}}}\,\Delta T^{(E=0)} \tag{7.69}$$

Note that for E going to the critical value $E_c = \frac{1}{2\pi}$, the lifetime becomes infinite. In particular we get a completely flat profile. This has to be traced back to the fact that open strings become effectively tensionless in this limit, [100], so we correctly recover the result that the D-brane is stable. This configuration should correspond to a background of fundamental strings stretched along the E–field direction, with closed strings completely decoupled.

7.7 Adding a transverse E–field

In this section we study the time dynamics of a D–brane with transverse E–field. We will do this in two steps. First we will write down coordinates and momenta operators corresponding to the oscillators of the discrete diagonal basis and look at the profile of the lump solution with respect to them. Next we will determine the open string time profile of the lump solution by projecting it onto the center of mass coordinates. Since the solutions with E–field are equivalent to euclidian solutions with imaginary B–field, before proceeding further, we will first give a brief summary of the construction of lump solutions in VSFT with transverse B–field.

7.7.1 Lump solutions with B field

The solitonic lump solutions in VSFT in the presence of a constant transverse B field were determined in [66, 63, 67]. The $*$ product is defined as follows

$$ {}_{123}\langle V_3|\Psi_1\rangle_1|\Psi_2\rangle_2 = {}_3\langle \Psi_1 *_m \Psi_2| \tag{7.70} $$

where the 3-string vertex V_3, with a constant B field turned on along the 24^{th} and 25^{th} directions (in view of the D-brane interpretation, these directions are referred to as transverse), is

$$ |V_3\rangle = |V_{3,\perp}\rangle \otimes |V_{3,||}\rangle. \tag{7.71} $$

$|V_{3,||}\rangle$ corresponds to the tangential directions while $|V_{3,\perp}\rangle$ is obtained from [42] by passing to zero modes oscillator basis and integrating over transverse momenta, see [66, 63, 67]

$$ |V_{3,\perp}\rangle = \frac{\sqrt{2\pi b^3\Delta}}{A^2(4a^2+3)}\exp\left[\frac{1}{2}\sum_{r,s=1}^{3}\sum_{N,M\geq 0} a_M^{(r)\alpha\dagger}\mathcal{V}^{rs}_{\alpha\beta,MN}a_N^{(r)\beta\dagger}\right]|0\rangle\otimes|\Omega_{b,\theta}\rangle_{123}. \tag{7.72} $$

In the following we will set $\alpha,\beta=1,2$ for simplicity of notation. $|\Omega_{b,\theta}\rangle$ is the vacuum with respect to the zero mode oscillators

$$ a_0^{(r)\alpha} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} - i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}, \qquad a_0^{(r)\alpha\dagger} = \frac{1}{2}\sqrt{b}\hat{p}^{(r)\alpha} + i\frac{1}{\sqrt{b}}\hat{x}^{(r)\alpha}. \tag{7.73} $$

$\mathcal{V}^{rs}_{\alpha\beta,MN}$ are the Neumann coefficients with zero modes in a constant B field background, which are symmetric under simultaneous exchange of all the three pairs of indices and cyclic in the string label indices (r,s) where $r,s=4$ is identified with $r,s=1$. Moreover $\Delta=\sqrt{\mathrm{Det}G}$, $G_{\alpha\beta}$ being the open string metric along the transverse directions (7.126). We have also introduced the notations

$$ A = V_{00}+\frac{b}{2}, \qquad a = -\frac{\pi^2}{A}|B|. \tag{7.74} $$

The lump solution is given by

$$ |S\rangle = |S_{||}\rangle\otimes\mathcal{N}\exp\left(-\frac{1}{2}\sum_{M,N\geq 0}a_M^{\alpha\dagger}\mathcal{S}_{\alpha\beta,MN}a_N^{\beta\dagger}\right)|0\rangle\otimes|\Omega_{b,\theta}\rangle, \tag{7.75} $$

where

$$\mathcal{N} = \frac{A^2(3+4a^2)}{\sqrt{2\pi b^3}(\mathrm{Det}G)^{\frac{1}{4}}}\mathrm{Det}(\mathcal{I}-\mathcal{X})^{\frac{1}{2}}\mathrm{Det}(\mathcal{I}+\mathcal{T})^{\frac{1}{2}}, \tag{7.76}$$

and

$$\mathcal{X} = C'\mathcal{V}^{11}, \qquad \mathcal{T} = C'\mathcal{S}, \qquad C' = (-1)^N\delta_{NM} \tag{7.77}$$

In (7.75) $|S_{||}\rangle$ corresponds to the longitudinal part of the lump solution and it is a zero momentum sliver.

In order for (7.75) to satisfy the projector equation, $\mathcal{T}$ and $\mathcal{X}$ should satisfy the relation[6]

$$(\mathcal{T}-1)(\mathcal{X}\mathcal{T}^2 - (\mathcal{I}+\mathcal{X})\mathcal{T} + \mathcal{X}) = 0. \tag{7.78}$$

In the above formulae the α, β, N, M indices are implicit. This equation is solved by $\mathcal{T}_0$, $1/\mathcal{T}_0$ and 1, where

$$\mathcal{T}_0 = \frac{1}{2\mathcal{X}}\left(1+\mathcal{X}-\sqrt{(1+3\mathcal{X})(1-\mathcal{X})}\right) \tag{7.79}$$

$\mathcal{T}=1$ gives the identity state, whereas the first and the second solutions give the lump and the inverse lump, respectively. In [5] it has been argued that, although the inverse lump solution was discarded in earlier works [38, 54], because of the bad behaviour of its eigenvalues in the oscillator basis, it is possible to make sense out of it by considering (7.78) as a relation between eigenvalues relative to twist definite eigenvectors. In particular, in the diagonal basis, the projector equation factorizes into the continuous and discrete contributions, which separately satisfy equation (7.78). Therefore, one can just invert (for example) the discrete eigenvalues of $\mathcal{T}$: dangerous $-$ signs under the square root in the energy densities of the solution are indeed avoided by counting the double multiplicity of these eigenvalues, which is required by twist invariance. See Appendix E for the spectroscopy of $\mathcal{X}$, and hence of $\mathcal{T}$.

7.7.2 Diagonal Coordinates and Momenta

In Appendix E τ–twist definite oscillators of the diagonal basis are introduced. Due to the structure of Neumann coefficients it is natural to define the twist matrix as τC, where $\tau = \sigma^3$ acts on space–time indices. In the following C–parity will be always understood as τC–parity. Now let's define the following coordinates and momenta operators in terms of the twist even and twist odd parts of the discrete spectrum, (E.39)

$$\hat{X}_{\xi_i} = \frac{i}{\sqrt{2}}(e_{\xi_i} - e^\dagger_{\xi_i}) \qquad \hat{Y}_{\xi_i} = \frac{i}{\sqrt{2}}(o_{\xi_i} - o^\dagger_{\xi_i}) \tag{7.80}$$

which are hermitian by definition and have the following eigenstates

$$|X_i\rangle = \frac{1}{\sqrt{\pi}}e^{-\frac{1}{2}X_i^2 - \sqrt{2}iX_i e^\dagger_{\xi_i} + \frac{1}{2}e^\dagger_{\xi_i}e^\dagger_{\xi_i}}|\Omega_{e_i}\rangle \tag{7.81}$$

[6]Here we limit ourselves to twist invariant projectors, but our analysis can be straightforwardly generalized to projectors of the kind [57]

$$|Y_i\rangle = \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}Y_i^2 - \sqrt{2}iY_i o^\dagger_{\xi_i} + \frac{1}{2} o^\dagger_{\xi_i} o^\dagger_{\xi_i}} |\Omega_{o_i}\rangle. \tag{7.82}$$

We made the assumption that the vacuum factorizes as

$$|0\rangle \otimes |\Omega_{b,\theta}\rangle = \prod_{i=1}^{2} \prod_{k} |\Omega_i(k)\rangle \otimes |\Omega_{e_i}\rangle \otimes |\Omega_{o_i}\rangle \tag{7.83}$$

where $|\Omega_i(k)\rangle$, $|\Omega_{e_i}\rangle$ and $|\Omega_{o_i}\rangle$ are vacua with respect to the continuous, the twist even discrete and twist odd discrete oscillators, respectively.

The explicit (X_i, Y_i) dependence of the lump state (7.75) can be obtained by projecting it onto the eigenstates $|X_i, Y_i\rangle$. After re-writing (7.75) in terms of the diagonal basis oscillators and performing the projection (see Appendix E), it follows

$$\langle X_i, Y_i|S\rangle = \frac{1}{\pi^2[1+t_d(\eta_1)][1+t_d(\eta_2)]} \exp\frac{1}{2}\left[\frac{t_d(\eta_1)-1}{t_d(\eta_1)+1}(X_1^2+Y_1^2) \right.$$
$$\left. + \frac{t_d(\eta_2)-1}{t_d(\eta_2)+1}(X_2^2+Y_2^2)\right] |S\rangle_c \otimes |S_{||}\rangle. \tag{7.84}$$

$|S\rangle_c$ is given by (E.44) with only continuous spectrum oscillators and $t_d(\eta_i) = e^{-|\eta_i|}$ are the discrete eigenvalues of $\mathcal{T}$ corresponding to the eigenvalue $\xi(\eta_i)$ of the operator $C'\mathcal{U}$.

In (7.84) the directions α, β are completely mixed. As a matter of fact, it is not apparent at this stage which of these variables (X_i, Y_i) contain the information about the center of mass time dependence of the lump. To make this clear let's recall the non-diagonal basis oscillators and write the coordinates and the momenta operators as

$$\hat{X}^\alpha_N = \frac{i}{\sqrt{2}}(a^\alpha_N - a^{\alpha\dagger}_N) \qquad \hat{P}^\alpha_N = \frac{1}{\sqrt{2}}(a^\alpha_N + a^{\alpha\dagger}_N). \tag{7.85}$$

In order to get the relation between these operators and the corresponding diagonal operators we have defined above, we need to re-write the diagonal basis oscillators in terms of the non-diagonal ones. In doing so, one has to be careful about taking the complex conjugate of the eigenstates, as we are dealing with hermitian rather then symmetric matrices. Taking this fact into account and using some results of Appendix E, we obtain

$$e_{\xi_i} = \frac{1}{\sqrt{2}} \sum_{N=0}^{\infty} (V_N^{(\xi_i)\alpha} + V_N^{(\bar{\xi}_i)\alpha}) a_{N,\alpha} \qquad e^\dagger_{\xi_i} = \frac{1}{\sqrt{2}} \sum_{N=0}^{\infty} (\bar{V}_N^{(\xi_i)\alpha} + \bar{V}_N^{(\bar{\xi}_i)\alpha}) a^\dagger_{N,\alpha} \tag{7.86}$$

$$o_{\xi_i} = \frac{-i}{\sqrt{2}} \sum_{N=0}^{\infty} (V_N^{(\xi_i)\alpha} - V_N^{(\bar{\xi}_i)\alpha}) a_{N,\alpha} \qquad o^\dagger_{\xi_i} = \frac{i}{\sqrt{2}} \sum_{N=0}^{\infty} (\bar{V}_N^{(\xi_i)\alpha} - \bar{V}_N^{(\bar{\xi}_i)\alpha}) a^\dagger_{N,\alpha} \tag{7.87}$$

and similar relations for the continuous spectrum oscillators. Hence, the diagonal coordinates and momenta can be written as

$$\hat{X}_{\xi_i} = \sqrt{2} \sum_{N=0}^{\infty} V_{2N}^{\xi_i,1} \hat{X}^1_{2N} + V_{2N+1}^{\xi_i,2} \hat{P}^2_{2N+1} \tag{7.88}$$

$$\hat{Y}_{\xi_i} = \sqrt{2}\sum_{N=0}^{\infty} V_{2N+1}^{\xi_i,1}\hat{P}_{2N+1}^1 - iV_{2N}^{\xi_i,2}\hat{X}_{2N}^2 \tag{7.89}$$

Now, to make the center of mass time dependence of the solution explicit, we need to extract the zero modes from these operators. Let's write the zero mode coordinate and momentum operators by introducing the b parameter as

$$\hat{X}_0^\alpha = \frac{i}{\sqrt{b}}(a_0^\alpha - a_0^{\alpha\dagger}) \qquad \hat{P}_0^\alpha = \frac{\sqrt{b}}{2}(a_0^\alpha + a_0^{\alpha\dagger}). \tag{7.90}$$

This gives

$$\hat{X}_{\xi_i} = \sqrt{2}\left[V_0^{\xi_i,1}\sqrt{\frac{2}{b}}X_0^1 + \sum_{n=1}^{\infty} V_{2n}^{\xi_i,1}\hat{X}_{2n}^1 + V_{2n-1}^{\xi_i,2}\hat{P}_{2n-1}^2\right], \tag{7.91}$$

$$\hat{Y}_{\xi_i} = \sqrt{2}\left[V_0^{\xi_i,2}\sqrt{\frac{2}{b}}X_0^2 + \sum_{n=1}^{\infty} V_{2n-1}^{\xi_i,1}\hat{P}_{2n-1}^1 - iV_{2n}^{\xi_i,2}\hat{X}_{2n}^2\right]. \tag{7.92}$$

Since our aim is to obtain the localization in time by making the inverse Wick rotation on direction 1, we see that it is X_{ξ_i} that contains the time coordinate, which we have to compare with the string center of mass time (see below).

7.7.3 Projection on the center of mass coordinates

In order to obtain the open string time profile of the lump solution, we need to project it onto the center of mass coordinates of the string. The center of mass position operator is given by

$$\hat{x}_{cm,\alpha} = \frac{i}{\sqrt{b}}(a_{0,\alpha} - a_{0,\alpha}^\dagger) \tag{7.93}$$

and its eigenstate is

$$|X_{CM}\rangle = \sqrt{\frac{2\Delta}{\pi b}}\, e^{-\frac{1}{b}x_\alpha x^\alpha - \frac{2}{\sqrt{b}}ix_\alpha a_0^{\alpha\,\dagger} + \frac{1}{2}a_{0,\alpha}^\dagger a_0^{\alpha\,\dagger}}|\Omega_{\theta,b}\rangle. \tag{7.94}$$

One can project the lump on this state to obtain the center of mass time profile. However, for reasons that will be clear later, we will first project on the Y_i momenta,

$$|\Lambda\rangle = \langle Y_1, Y_2|S\rangle = \frac{\mathcal{N}}{\pi\sqrt{[1+t_d(\eta_1)][1+t_d(\eta_2)]}}\exp\frac{1}{2}\left[\frac{t_d(\eta_1)-1}{t_d(\eta_1)+1}Y_1^2 + \frac{t_d(\eta_2)-1}{t_d(\eta_2)+1}Y_2^2\right]$$

$$\times\exp -\frac{1}{2}\left[e_{\xi_i}^\dagger e_{\xi_i}^\dagger t_d(\eta_i) + \int_{-\infty}^{\infty} dk a_i^\dagger(k)a_{i+1}^\dagger(-k)t_c(k)\right]|\Omega_e\rangle\otimes|\Omega_c\rangle\otimes|S_{||}\rangle. \tag{7.95}$$

Where we have used the notation

$$|\Omega_e\rangle = \prod_{i=1}^{2}|\Omega_{e_i}\rangle, \qquad |\Omega_c\rangle = \prod_{i=1}^{2}\prod_k|\Omega_i(k)\rangle. \tag{7.96}$$

Taking equation (7.87) and the corresponding relations for the continuous spectrum oscillators, equation (7.95) can be rewritten as

$$|\Lambda\rangle = \frac{\mathcal{N}}{\pi\sqrt{[1+t_d(\eta_1)][1+t_d(\eta_2)]}}\exp\frac{1}{2}\left[\frac{t_d(\eta_1)-1}{t_d(\eta_1)+1}Y_1^2+\frac{t_d(\eta_2)-1}{t_d(\eta_2)+1}Y_2^2\right]$$

$$\times\exp\left[-\frac{1}{2}a^\dagger_{0,\alpha}\hat{S}^{\alpha\beta}_{00}a^\dagger_{0,\beta}-a^\dagger_{0,\alpha}S^\alpha_0-\frac{1}{2}a^\dagger_{n,\alpha}\hat{S}^{\alpha\beta}_{nm}a^\dagger_{m,\beta}\right]|\hat{\Omega}_{b,\theta}\rangle\otimes|S_{||}\rangle, \tag{7.97}$$

where $|\hat{\Omega}_{b,\theta}\rangle = |\Omega_e\rangle\otimes|\Omega_c\rangle$ and

$$\hat{S}^{\alpha\beta}_{00} = \sum_{i=1}^{2}V_0^{(\xi_i^+)\alpha}\bar{V}_0^{(\xi_i^+)\beta}t_d(\eta_i)+\int_{-\infty}^{\infty}dkt(k)V_0^{i,\alpha}(k)\bar{V}_0^{i,\beta}(k) \tag{7.98}$$

$$S^\alpha_0 = \sum_{i=1}^{2}\sum_{n=1}\left[V_0^{(\xi_i^+)\alpha}\bar{V}_n^{(\xi_i^+)\beta}t_d(\eta_i)+\int_{-\infty}^{\infty}dkt(k)V_0^{i,\alpha}(k)\bar{V}_n^{i,\beta}(k)\right]a^\dagger_{n,\beta} = \hat{S}^{\alpha\beta}_{0n}a^\dagger_{n,\beta} \tag{7.99}$$

$$\hat{S}^{\alpha\beta}_{nm} = \sum_{i=1}^{2}(-1)^nV_n^{(\xi_i^+)\alpha}\bar{V}_m^{(\xi_i^+)\beta}t_d(\eta_i)+\int_{-\infty}^{\infty}dkt(k)(-1)^nV_n^{i,\alpha}(k)\bar{V}_m^{i,\beta}(k) \tag{7.100}$$

with $V_N^{(\xi_i^+)\alpha}$ being the twist even combination of the discrete eigenstates, see appendix E. Now let's project onto the center of mass coordinates

$$\langle X_{CM}|\Lambda\rangle = \frac{\mathcal{N}}{\pi\sqrt{[1+t_d(\eta_1)][1+t_d(\eta_2)]}}\exp\frac{1}{2}\left[\frac{t_d(\eta_1)-1}{t_d(\eta_1)+1}Y_1^2+\frac{t_d(\eta_2)-1}{t_d(\eta_2)+1}Y_2^2\right]$$

$$\times\sqrt{\frac{2\Delta}{\pi b}}\frac{1}{\sqrt{[1+s_1][1+s_2]}}\exp\frac{1}{b}\left[\frac{s_1-1}{s_1+1}x_1x^1+\frac{s_2-1}{s_2+1}x_2x^2+2i\sqrt{b}\left(\frac{S_{0,1}x^1}{1+s_1}+\frac{S_{0,2}x^2}{1+s_2}\right)\right]$$

$$\times\exp\left[-\frac{1}{2}a^\dagger_{n,\beta}\left(\hat{S}^{\alpha\beta}_{nm}-\frac{\hat{S}^\alpha_{n0,1}\hat{S}^{1\beta}_{0m}}{1+s_1}-\frac{\hat{S}^\alpha_{n0,2}\hat{S}^{2\beta}_{0m}}{1+s_2}\right)a^\dagger_{n,\beta}\right]|0\rangle\otimes|S_{||}\rangle \tag{7.101}$$

where

$$s_1 = 2\Delta[g_d^2(\eta_1,\eta_2)t_d(\eta_1)+g_d^2(\eta_2,\eta_1)t_d(\eta_2)]+\Delta\int_{-\infty}^{\infty}dk\ t_c(k)[g_c^2(k)+g_c^2(-k)),$$

$$s_2 = \Delta\int_{-\infty}^{\infty}dk\ t_c(k)[g_c^2(k)+g_c^2(-k)], \tag{7.102}$$

$t_c(k) = -e^{-\pi|k|/2}$ is the eigenvalue of $\mathcal{T}$ in the continuous spectrum, see Appendix E for the definition of the remaining terms which enter in the last two equations. The inverse Wick-rotation along direction 1 of (7.101) should give us a time-localized solution. It depends on two parameters, b and a, which can be expressed in terms of (η_1,η_2), through the eigenvalues equations (E.16) . Let's now take a look at every term in this solution and analyze it for different values of the such parameters.

In the Wick-rotated solution, to get time-localization, the term $-\frac{1}{b}\frac{s_1-1}{s_1+1}$ should be negative. We cannot achieve this using the conventional lump, since in this case $-1<$

$s_1 < 1$. To correct this, as anticipated, we need to invert one or two discrete eigenvalues, ($t_d(\eta_1)$ or/and $t_d(\eta_2)$). In this case one can easily show that $1 < s_1 < \infty$ and we get the desired behaviour. Given the possibility of inverting one or two eigenvalues, it might seem that there is some arbitrariness in our procedure. Actually there is none, since the cancelation of the potentially divergent terms when $b \to \infty$ (see below), requires the inversion of only one eigenvalue. In addition, time localization in small b regime requires the inversion of the eigenvalue of $\mathcal{T}$ corresponding to the greater between η_1 and η_2 (η_2 in our conventions). From now on we will then consider a solution in which $t_d(\eta_2)$ is inverted, i.e. $t_d(\eta_2) \to t_d^{-1}(\eta_2)$.

Next, look at the term $\frac{s_2-1}{s_2+1} x_2 x^2$. Due to the $\langle Y_1, Y_2|$ projection, it gets a contribution only from the continuous spectrum, which is always negative and in the range $(-1, 0)$. As a result, this second term is always negative and gives localization in the transverse space direction.

Now we would like to point out some facts about the two parameters on which our solution depends. Previously, in the $E = 0$ case, it has been pointed out that the inverse of the parameter b, for large b, plays the role of Sen's $\tilde{\lambda}$ near zero. Here again we can repeat the same argument. Note however that in taking b to infinity we should keep a vanishing, see (7.74), since we cannot overcome the critical value $|B|_c = \frac{1}{2\pi}$. For this reason the result of taking $b \to \infty$ is insensible of the value of the E-field, making this limit completely commutative.

As it is justified in Appendix E, the proper way to send b to ∞ is to take $\eta_1 \approx \eta_2 \to \infty$ keeping $\eta_1 < \eta_2$. In this case one can easily see that

$$s_1 \approx \eta_2 \log \eta_1 \eta_2 + t_c(k_0 \approx 0), \qquad s_2 \approx t_c(k_0 \approx 0) \tag{7.103}$$

with k_0 as defined in Appendix E. Note that $t_c(k_0 \approx 0) = -1$. This is so because the E–field cannot scale to infinity due to existence of critical value. Then it follows

$$\lim_{\eta_1, \eta_2 \to \infty} \frac{s_1 - 1}{s_1 + 1} = 1, \qquad \lim_{\eta_1, \eta_2 \to \infty} \frac{s_2 - 1}{s_2 + 1} = -\infty \tag{7.104}$$

As justified in Appendix E, in this limit $\hat{S}_{n0}^{\alpha\beta(c)} = 0$ so that the oscillating term in (7.101) receives a contribution only from the discrete part. It is also pointed out that the discrete contribution vanishes except for $\alpha = \beta = 1$, which is the only non trivial contribution to the oscillating term. Moreover, we have

$$\lim_{\eta_1, \eta_2 \to \infty} \frac{\Delta \hat{S}_{n0}^{11}}{s_1 + 1} = (-1)^n \lim_{\eta_1, \eta_2 \to \infty} \frac{1}{2\sqrt{\log \eta_1 \eta_2}} = 0, \tag{7.105}$$

Therefore, the oscillating term in (7.101) vanishes when $b \to \infty$.

Now let's consider the non-zero mode terms, i.e, the last line in (7.101). In the $b \to \infty$ limit it is clear that $V_n^{(\xi_i^+)\alpha}$ vanishes for $\alpha = 2$ and $n \geq 1$. Therefore, the contribution of the discrete spectrum to $\hat{S}^{\alpha\beta}_{nm}$ is zero for α or $\beta = 2$ and $n, m \geq 1$. However, for $\alpha = \beta = 1$ this is not true and there are potentially divergent contributions from the discrete spectrum. We are now going to show that these divergences cancel and the expression

$$\check{S}^{11}_{nm} = \hat{S}^{11(c)}_{nm} + \hat{S}^{11(d)}_{nm} - \frac{\hat{S}^{1(d)}_{n0,1}\hat{S}^{11(d)}_{0m}}{1+s_1}. \tag{7.106}$$

is finite when $b \to \infty$.

To this end we notice that, inverting only $t_d(\eta_2)$ but taking both η_1 and η_2 to infinity, the different terms which enter in the above expression have the following behaviors

$$\begin{aligned}
1 + s_1 &\approx \Delta t_d^{-1}(\eta_2) V_0^{(\xi_2^+),1} \bar{V}_0^{(\xi_2^+),1}, \\
\hat{S}^{1(d)}_{n0,1} &\approx \Delta(-1)^n t_d^{-1}(\eta_2) V_n^{(\xi_2^+),1} \bar{V}_0^{(\xi_2^+),1}, \\
\hat{S}^{11(d)}_{n0} &\approx (-1)^n t_d^{-1}(\eta_2) V_n^{(\xi_2^+),1} \bar{V}_0^{(\xi_2^+),1}, \\
\hat{S}^{11(d)}_{nm} &\approx (-1)^n t_d^{-1}(\eta_2) V_n^{(\xi_2^+),1} \bar{V}_m^{(\xi_2^+),1},
\end{aligned} \tag{7.107}$$

Note that $t_d^{-1}(\eta_2) = e^{|\eta_2|}$ gives a divergent contribution as $\eta_2 \to \infty$. However, using these results in eq.(7.106), it is easy to see that the divergent terms cancel and we are left with $\check{S}^{11}_{nm} = \hat{S}^{11(c)}_{nm}$. This, combined with the fact that for $\alpha = 2$ or $\beta = 2$ we have $\hat{S}^{\alpha\beta(d)}_{NM} = 0$, leads us to the conclusion that $\check{S}^{\alpha\beta}_{nm} = \hat{S}^{\alpha\beta(c)}_{nm} + O\left(\frac{1}{b}\right)$. It is also verified in Appendix E that $\hat{S}^{11(c)}_{nm} = \hat{S}^{22(c)}_{nm} = S_{nm}$.

This also show that is not possible to invert both the discrete eigenvalues and obtain the same cancelation. Indeed, if we invert both, the term $\hat{S}^{1(d)}_{n0,1}\hat{S}^{11(d)}_{0m}$ contains mixed terms like $[t^{-1}(\eta_1)V_n^{(\xi_1^+),1}\bar{V}_0^{(\xi_1^+),1}][t^{-1}(\eta_2)V_n^{(\xi_2^+),1}\bar{V}_0^{(\xi_2^+),1}]$, for which we cannot find a counter term in $\hat{S}^{11(d)}_{nm}$ to cancel it. As a result we will not be able to get a regular time and space localized solution, since these terms diverge in the limit $\eta_1, \eta_2 \to \infty$.

After all these remarks, we can write the space-time localized solution in the $b \to \infty$ limit as

$$\lim_{b\to\infty} \langle X_{CM}|\Lambda\rangle_{Wick} = N(Y_1, Y_2) \lim_{b\to\infty} e^{-\frac{\Delta}{b}(x^0)^2} e^{-\epsilon(b)(x^2)^2}|S\rangle \tag{7.108}$$

where $|S\rangle$ is the space-time independent VSFT solution (the sliver). Note that time dependence completely disappears in this limit . A remark is in order for the quantity $\epsilon(b)$ this number is given by, see (7.101)

$$\epsilon(b) = \frac{\Delta}{b}\frac{s_2 - 1}{s_2 + 1} \tag{7.109}$$

a numerical analysis shows that this becomes vanishing as $b \to \infty$. One can indeed easily check (numerically) that the $\frac{1}{b}$ correction to $\frac{s_2(b)-1}{s_2(b)+1}$ diverges. This in turn implies that the loss of time dependence is accompanied by loss of transverse space dependence, giving a resulting zero momentum state (the D25–sliver). Therefore, taking b to infinity is like sitting at the original unstable vacuum (the D25–brane), which is the same situation as setting Sen's $\tilde{\lambda}$ to zero.

Another remark we would like to make is about small b limit, which we can get by taking $\eta_1 \to 0$ and keeping η_2 finite. Given that the large b limit corresponds to Sen's $\tilde{\lambda}$ near zero (i.e it represents the unstable vacuum), it is natural to think that the small b limit corresponds to $\tilde{\lambda}$ near $\frac{1}{2}$ (or the stable vacuum). As a matter of fact, taking this limit of b one gets the 0 state, which is also obtained in the $x_0 \to \infty$ limit and corresponds to the stable vacuum to which the D-brane decays. This can be seen by noting that, in this case, $V_0^\alpha(k) \to 0$, whereas $V_n^\alpha(k)$ for $n \geq 1$ have a finite nonvanishing limit. As a result s_1 do not get a contribution from the continuous spectrum and $s_2 = 0$. Then, it follows

$$-\frac{\Delta}{b}\frac{s_1 - 1}{s_1 + 1} \approx -\frac{\Delta}{\eta_1}\left(\left|\frac{s_1 - 1}{s_1 + 1}\right|\right), \qquad \frac{\Delta}{b}\frac{s_2 - 1}{s_2 + 1} \approx -\frac{\Delta}{\eta_1} \tag{7.110}$$

where we have used ($b \approx \eta_1$), ($s_1 \approx 1 + O(\sqrt{\eta_1})$) in the limit $\eta_1 \to 0$ and η_2 finite. These are results one can easily obtain from appendix E. For $\Delta \neq 0$ both of these terms gives a negative infinity in the exponent and suppress everything in front to give us the 0 state which corresponds to the stable vacuum. However, the case $\Delta = 0$ should be handled with care. In this case, one can send Δ and η_1 to zero, in such a way that the ratio $\frac{\Delta}{\eta_1}$ remains finite. As a result the time dependence will be lost while the solution is still space localized. One should compare this with the time independent solution obtained when we send Sen's $\tilde{\lambda}$ to $\frac{1}{2}$ and, at the same time, tune the E–field to its critical value, [36], obtaining a static fundamental strings background.

7.8 A proposal for macroscopic F–strings

A rolling tachyon describes in various languages (effective field theory, BCFT, SFT) the decay of unstable D-branes . It is by now clear that the final product of a brane decay is formed by massive closed string states. However it has been shown that, in the presence of a background electric field also (macroscopic) fundamental strings appear as final products of a brane decay. Now, since our aim is to be able to describe a brane decay in the framework of VSFT we must show that such fundamental strings exist as solutions of VSFT. In this last section we want to present some evidence that such solutions do exist. We have already said that fundamental strings carry Kalb–Ramond charge, we will indeed see that such solutions can be properly defined only in a $B_{\mu\nu}$ background.

7.8.1 Constructing new solutions

First of all we would like to show how qualitatively new solutions to (4.6) can be constructed by accretion of infinite many lumps. Let us start from a lump solution representing a D0–brane as introduced in the previous section: it has a Gaussian profile in all space directions, the form of the string field – let us denote it $|\Xi_0'\rangle$ – will be the same as (4.12) with S replaced by S', while the $*$–product will be determined by the primed three strings vertex (2.47). Let us pick one particular space direction, say the α–th. For simplicity in the following we will drop the corresponding label from the coordinate $\hat{x}^\alpha$, momenta $\hat{p}^\alpha$ and oscillators a^α along this direction. Next we need the same solution displaced by an amount s in the positive x direction (x being the eigenvalue of $\hat{x}$). The appropriate solution has been constructed by Rastelli, Sen and Zwiebach, [30]:

$$|\Xi_0'(s)\rangle = e^{-is\hat{p}}|\Xi_0'\rangle \tag{7.111}$$

It satisfies $|\Xi_0'(s)\rangle * |\Xi_0'(s)\rangle = |\Xi_0'(s)\rangle$. Eq.(7.111) can be written explicitly as

$$|\Xi_0'(s)\rangle = \mathcal{N}' e^{-\frac{s^2}{2b}(1-S_{00}')} \exp\left(-\frac{is}{\sqrt{b}}((1-S')\cdot a^\dagger)_0\right) \exp\left(-\frac{1}{2}a^\dagger\cdot S'\cdot a^\dagger\right)|\Omega_b\rangle \tag{7.112}$$

where $((1-S_{00}')\cdot a^\dagger)_0 = \sum_{N=0}^\infty ((1-S')_{0N}a_N^\dagger)$ and $a^\dagger\cdot S'\cdot a^\dagger = \sum_{N,M=0}^\infty a_N^\dagger S_{NM}' a_M^\dagger$; $\mathcal{N}'$ is the $|\Xi_0'\rangle$ normalization constant. Moreover one can show that

$$\langle\Xi_0'(s)|\Xi_0'(s)\rangle = \langle\Xi_0'|\Xi_0'\rangle \tag{7.113}$$

The meaning of this solution is better understood if we make its space profile explicit by contracting it with the coordinate eigenfunction

$$|\hat{x}\rangle = \left(\frac{2}{\pi b}\right)^{\frac{1}{4}} \exp\left(-\frac{x^2}{b} - i\frac{2}{\sqrt{b}}a_0^\dagger x + \frac{1}{2}a_0^\dagger a_0^\dagger\right)|\Omega_b\rangle \tag{7.114}$$

The result is

$$\langle\hat{x}|\Xi_0'(s)\rangle = \left(\frac{2}{\pi b}\right)^{\frac{1}{4}} \frac{\mathcal{N}'}{\sqrt{1+S_{00}'}} e^{-\frac{1-S_{00}'}{1+S_{00}'}\frac{(x-s)^2}{b}} e^{-\frac{2i}{\sqrt{b}}\frac{x-s}{1+S_{00}'}S_{0m}'a_m^\dagger} e^{-\frac{1}{2}a_n^\dagger W_{nm} a_m^\dagger}|0\rangle \tag{7.115}$$

where $W_{nm} = S_{nm}' - \frac{S_{n0}'S_{0m}'}{1+S_{00}'}$. It is clear that (7.115) represents the same Gaussian profile as $|\Xi_0'\rangle = |\Xi_0'(0)\rangle$ shifted away from the origin by s.

It is important to remark now that two such states $|\Xi_0'(s)\rangle$ and $|\Xi_0'(s')\rangle$ are $*$–orthogonal and bpz–orthogonal provided that $s \neq s'$. For we have

$$|\Xi_0'(s)\rangle * |\Xi_0'(s')\rangle = e^{-\mathcal{C}(s,s')}|\Xi_0'(s,s')\rangle \tag{7.116}$$

where the state $|\Xi_0'(s,s')\rangle$ becomes proportional to $|\Xi_0'(s)\rangle$ when $s = s'$ and needs not be explicitly written down otherwise; while

$$\mathcal{C}(s,s') = -\frac{1}{2b}\left[(s^2+s'^2)\left(\frac{T'(1-T')}{1+T'}\right)_{00} + ss'\left(\frac{(1-T')^2}{1+T'}\right)_{00}\right] \tag{7.117}$$

The quantity $\left(\frac{T'(1-T')}{1+T'}\right)_{00}$ can be evaluated by using the basis of eigenvectors of X' and T', see chapter 2 and appendix B:

$$\left(\frac{T'(1-T')}{1+T'}\right)_{00} \tag{7.118}$$

$$= 2\int_0^\infty dk\,(V_0(k))^2 \frac{t(k)(1-t(k))}{1+t(k)} + \left(V_0^{(\xi)}V_0^{(\xi)} + V_0^{(\bar{\xi})}V_0^{(\bar{\xi})}\right)\frac{e^{-|\eta|}(1-e^{-|\eta|})}{1+e^{-|\eta|}}$$

The variable k parameterizes the continuous spectrum and $V_0^{(k)}$ is the relevant component of the continuous basis. The modulus 1 numbers ξ and $\bar{\xi}$ parameterize the discrete spectrum and $V_0^{(\xi)}$, $V_0^{(\bar{\xi})}$ are the relevant components of the discrete basis . The discrete spectrum part of the RHS of (7.118) is just a number. Let us concentrate on the continuous spectrum contribution. We have $t(k) = -\exp(-\frac{\pi|k|}{2})$. Near $k = 0$, $V_0(k) \sim \frac{1}{2}\sqrt{\frac{b}{2\pi}}$ and the integrand $\sim -\frac{b}{2\pi^2}\frac{1}{k}$, therefore the integral diverges logarithmically, a singularity we can regularize with an infrared cutoff ϵ. Taking the signs into account we find that the RHS of (7.118) goes like $\frac{b}{2\pi^2}\log\epsilon$ as a function of the cutoff. Similarly one can show that $\left(\frac{(1-T')^2}{1+T'}\right)_{00}$ goes like $-\frac{b}{\pi^2}\log\epsilon$. Since for $s \neq s'$, $s^2 + s'^2 > 2ss'$, we can conclude that $\mathcal{C}(s,s') \sim -c\log\epsilon$, where c is a positive number. Therefore, when we remove the cutoff, the factor $e^{-\mathcal{C}(s,s')}$ vanishes, so that (7.116) becomes a $*$–orthogonality relation. Notice that the above logarithmic singularities in the two pieces in the RHS of (7.118) neatly cancel each other when $s = s'$ and we get the finite number

$$\mathcal{C}(s,s) = -\frac{s^2}{2b}(1 - S'_{00})$$

In conclusion we can write

$$|\Xi'_0(s)\rangle * |\Xi'_0(s')\rangle = \hat{\delta}(s,s')|\Xi'_0(s)\rangle \tag{7.119}$$

where $\hat{\delta}$ is the Kronecker (not the Dirac) delta function.

Similarly one can prove that

$$\begin{aligned}&\langle\Xi'_0(s')|\Xi'_0(s)\rangle \\ &= \frac{\mathcal{N}^2}{\sqrt{\det(1-S'^2)}}\, e^{-\frac{s^2}{b}(1-S'_{00})}\, e^{\frac{1}{2b}\left[(s^2+s'^2)\left(\frac{S'(1-S')}{1+S'}\right)_{00}+2ss'\left(\frac{1-S'}{1+S'}\right)_{00}\right]}\end{aligned} \tag{7.120}$$

We can repeat the same argument as above and conclude that

$$\langle\Xi'_0(s')|\Xi'_0(s)\rangle = \hat{\delta}(s,s')\,\langle\Xi'_0|\Xi'_0\rangle \tag{7.121}$$

After the above preliminaries, let us consider a sequence $s_1, s_2, \ldots$ of distinct real numbers and the corresponding sequence of displaced D0–branes $|\Xi'_0(s_n)\rangle$. Due to the property (7.119) also the string state

$$|\Lambda\rangle = \sum_{n=1}^{\infty}|\Xi'_0(s_n)\rangle \tag{7.122}$$

is a solution to (4.6): $|\Lambda\rangle * |\Lambda\rangle = |\Lambda\rangle$. To figure out what it represents let us study its space profile. To this end we must sum all the profiles like (7.115) and then proceed to a numerical evaluation. In order to get a one dimensional object, we render the sequence $s_1, s_2, \ldots$ dense, say, in the positive x–axis so that we can replace the summation with an integral. The relevant integral is

$$\int_0^\infty ds \exp\left[-\alpha(x-s)^2 - i\beta(x-s)\right] = \frac{\sqrt{\pi}}{2\sqrt{\alpha}}\left(e^{-\frac{\beta^2}{4\alpha}}\left(1 + \mathrm{Erf}\left(\frac{i\beta}{2\sqrt{\alpha}} + \sqrt{\alpha}x\right)\right)\right) \tag{7.123}$$

where Erf is the error function and

$$\alpha = \frac{1}{b}\frac{1-S'_{00}}{1+S'_{00}}, \qquad \beta = \frac{2}{\sqrt{b}}\frac{S'_{0m}a^\dagger_m}{1+S'_{00}}$$

Of course (7.123) is a purely formal expression, but it becomes meaningful in the $\alpha' \to 0$ limit. As usual, [64], we parameterize this limit with a dimensionless parameter ϵ and take $\epsilon \to 0$. Using the results of appendix B, one can see that $\alpha \sim 1/\epsilon$, $\beta \sim 1/\sqrt{\epsilon}$, so that $\beta/\sqrt{\alpha}$ tends to a finite limit. Therefore, in this limit, we can disregard the first addend in the argument of Erf. Then, up to normalization, the space profile of $|\Lambda\rangle$ is determined by

$$\frac{1}{2}\left(1 + \mathrm{Erf}(\sqrt{\alpha}x)\right) \tag{7.124}$$

In the limit $\epsilon \to 0$ this factor tends to a step function valued 1 in the positive real x–axis and 0 in the negative one. Of course a similar result can be obtained numerically to any degree of accuracy by using a dense enough discrete $\{s_n\}$ sequence.

Another way of getting the same result is to use the recipe of [64] first on (7.115). In this way the middle exponential disappears, while the first exponential is regularized by hand (remember that $S'_{00} \to -1$ as $\epsilon \to 0$), so we replace S'_{00} by a parameter $\mathfrak{s}$ and keep it $\neq -1$. Now it is easy to sum over s_n. Again we replace the summation by an integration and see immediately that the space profile becomes the same as (7.124).

Let us stress that the derivation of the space profile in the low energy regime we have given above is far from rigorous. This is due to the very singular nature of the lump in this limit, first pointed out by [64]. A more satisfactory derivation will be provided in the next section after introducing a background B field.

In summary, the state $|\Lambda\rangle$ is a solution to (4.6), which represents, in the low energy limit, a one–dimensional object with a constant profile that extends from the origin to infinity in the x–direction. Actually the initial point could be any finite point of the x–axis, and it is not hard to figure out how to construct a configuration that extend from $-\infty$ to $+\infty$. How should we interpret these condensate of D0–branes? In the absence of supersymmetry it is not easy to distinguish between D–strings and F–strings (see, for instance, [101] for a comparison), however in the last section we will provide some evidence that the one–dimensional solutions of the type $|\Lambda\rangle$ can be interpreted as fundamental strings. This kind of objects are very well–known in string theory as classical solutions, [102, 103, 104, 105, 106], see also [107, 108, 109]. For the time being let us notice that, due to (7.121),

$$\langle\Lambda|\Lambda\rangle = \sum_{n,m=1}^{\infty} \langle\Xi'_0(s_n)|\Xi'_0(s_m)\rangle = \sum_{n=1}^{\infty}\langle\Xi'_0|\Xi'_0\rangle \tag{7.125}$$

It follows that the energy of the solution is infinite. Such an (unnormalized) infinity is a typical property of fundamental string solutions, see [102].

7.8.2 An improved construction

In this section we would like to justify some of the passages utilized in discussing the space profile of the fundamental string solution . The problems of the previous subsection are linked to the well–known singularity of the lump space profile, [64], which arises in the low energy limit ($\epsilon \to 0$) and renders some of the manipulations rather slippery. The origin of this singularity is the denominator $1 + S'_{00}$ that appears in many exponentials. Since, when $\epsilon \to 0$, $S'_{00} \to -1$ the exponentials are ill–defined because the series expansions in $1/\epsilon$ are. The best way to regularize them is to introduce a constant background B–field, [40, 41, 42]. The relevant formulas can be found in [66]. For the purpose of this paper we introduce a B field along two space directions, say x and y (our aim is to regularize the solution in the x direction, but, of course, there is no way to avoid involving in the process another space direction).

Let us use the notation x^α with $\alpha = 1, 2$ to denote x, y and let us denote

$$G_{\alpha\beta} = \Delta \delta_{\alpha\beta}, \qquad \Delta = 1 + (2\pi B)^2 \tag{7.126}$$

the open string metric. As is well–known, as far as lump solutions are concerned, there is an isomorphism of formulas with the ordinary case by which X', S', T' are replaced, respectively, by $\mathcal{X}, \mathcal{S}, \mathcal{T}$, which explicitly depend on B. One should never forget that the latter matrices involve two space directions. We will denote by $|\hat{\Xi}_0\rangle$ the D0–brane solution in the presence of the B field.

Without writing down all the details, let us see the significant changes. Let us replace formula (7.111) by

$$|\hat{\Xi}_0(\{s^\alpha\})\rangle = e^{-is^\alpha \hat{p}_\alpha} |\hat{\Xi}_0\rangle \tag{7.127}$$

It satisfies $|\hat{\Xi}_0(s)\rangle * |\hat{\Xi}_0(s)\rangle = |\hat{\Xi}_0(s)\rangle$ and $\langle \hat{\Xi}_0(s)|\hat{\Xi}_0(s)\rangle = \langle \hat{\Xi}_0|\hat{\Xi}_0\rangle$. Instead of (7.114) we have

$$|\{\hat{x}^\alpha\}\rangle = \left(\frac{2\Delta}{\pi b}\right)^{\frac{1}{2}} \exp\left[\left(-\frac{x^\alpha x^\beta}{b} - i\frac{2}{\sqrt{b}} a_0^{\alpha\dagger} x^\beta + \frac{1}{2} a_0^{\alpha\dagger} a_0^{\beta\dagger}\right) G_{\alpha\beta}\right] |\Omega_b\rangle \tag{7.128}$$

Next we have

$$\begin{aligned} \langle\{\hat{x}^\alpha\}|\hat{\Xi}_0(s)\rangle \; = \; & \left(\frac{2\Delta}{\pi b}\right)^{\frac{1}{2}} \frac{\hat{\mathcal{N}}}{\sqrt{\det(1+\mathcal{S}_{00})}} \exp\left[-\frac{1}{b}(x^\alpha - s^\alpha)\left(\frac{1-\mathcal{S}_{00}}{1+\mathcal{S}_{00}}\right)_{\alpha\beta}(x^\beta - s^\beta)\right. \\ & \left. - \frac{2i}{\sqrt{b}}(x^\alpha - s^\alpha)(1+\mathcal{S}_{00})_{\alpha\beta} \mathcal{S}_{0m}{}^\beta{}_\gamma \, a_m^{\gamma\dagger}\right] \exp\left[-\frac{1}{2} a_n^{\alpha\dagger} \mathcal{W}_{nm,\alpha\beta} a_m^{\beta\dagger}\right] |0\rangle \end{aligned} \tag{7.129}$$

where $\det(1 + \mathcal{S}_{00})$ means the determinant of the 2x2 matrix $(1 + \mathcal{S}_{00})_{\alpha\beta}$ and

$$\mathcal{W}_{nm,\alpha\beta} = \mathcal{S}_{nm,\alpha\beta} - \mathcal{S}_{n0,\alpha}{}^\gamma \left(\frac{1}{1+\mathcal{S}_{00}}\right)_{nm,\gamma\delta} \mathcal{S}_{0m}{}^\delta \beta \tag{7.130}$$

The state we start from, i.e. $|\hat{\Xi}_0(s)\rangle$, and the relevant space profile, are obtained by setting $s^1 = s$ and $s^2 = 0$ in the previous formulas.

Next we have an analog of (7.116) with $\mathcal{C}(s, s')$ replaced by

$$\hat{\mathcal{C}}(s, s') = -\frac{1}{2b}(s^2 + s'^2)\left(\frac{\mathcal{T}(1-\mathcal{T})}{1+\mathcal{T}}\right)_{00,11} - \frac{ss'}{2b}\left(\frac{(1-\mathcal{T})^2}{1+\mathcal{T}}\right)_{00,11} \tag{7.131}$$

Proceeding in the same way as before we can prove the analog of eq.(7.119). By using the spectral representation of appendix E one can show that $\hat{\mathcal{C}}$ picks up a logarithmic singularity unless $s = s'$. In a similar way one can prove the analog of (7.121).

Now let us discuss the properties of

$$|\hat{\Lambda}\rangle = \sum_{n=1}^{\infty} |\hat{\Xi}_0(s_n)\rangle$$

in the low energy limit. We refer to (7.129) with $s^1 = s$ and $s^2 = 0$. The fundamental difference between this formula and (7.115) is that in the low energy limit $\mathcal{S}_{00,\alpha\beta}$ becomes diagonal and takes on a value different from -1. More precisely

$$\mathcal{S}_{00,\alpha\beta} \rightarrow \frac{2|a|-1}{2|a|+1} G_{\alpha\beta}, \qquad a = -\frac{\pi^2}{V_{00} + \frac{b}{2}} B$$

see [63]. Therefore the $1 + \mathcal{S}_{00}$ denominators in (7.129) are not dangerous any more. Similarly one can prove that in the same limit $\mathcal{S}_{0n} \rightarrow 0$. Moreover the ϵ–expansions about these values are well–defined. Therefore the space profile we are interested in is

$$\sim \exp[-\frac{\mu}{b}(x+s)^2 - \frac{\mu}{b}(y)^2] \exp[-\frac{1}{2} a_n^{\alpha\dagger} \mathcal{S}_{nm,\alpha\beta} a_m^{\beta\dagger}]|0\rangle \tag{7.132}$$

with a finite normalization factor and $\mu = \frac{2|a|-1}{2|a|+1}\Delta$. Now one can safely integrate s and obtain the result illustrated in section 3. This also shed light on how the resulting state couples to the $B_{\mu\nu}$ field. Indeed the length of this one dimensional objects is measured with the open string metric (7.126), in other words the B–field couples to the string by "stretching" it.

7.8.3 Fundamental strings

In this section we would like to discuss the properties of the Λ solutions we found in the previous sections. In order to justify the claim we made that they represent fundamental strings, in the sequel we show that they are still solutions if we attach them to a D–brane. To this end let us pick $|\Lambda\rangle$ as given by (7.122) with $s_n > 0$ for all n's. Now let us consider a D24–brane with the only transverse direction coinciding with the x–axis and centered at $x = 0$. The corresponding lump solution has been introduced at the end of section 2 (case $k = 24$). Let us call it $|\Xi'_{24}\rangle$. Due to the particular configuration chosen, it is easy to prove that $|\Xi'_{24}\rangle + |\Lambda\rangle$ is still a solution to (4.6). This is due to the fact that $|\Xi'_{24}\rangle$

is $*$-orthogonal to the states $|\Xi_0'(s_n)\rangle$ for all n's. To be even more explicit we can study the space profile of $|\Xi_{24}'\rangle + |\Lambda\rangle$, assuming the sequence s_n to become dense in the positive x–axis. Using the previous results it is not hard to see that the overall configuration is a Gaussian centered at $x = 0$ in the x direction (the D24-brane) with an infinite prong attached to it and extending along the positive x–axis. The latter has a Gaussian profile in all space directions except x.

We remark that the condition $s_n > 0$ for all n's is important because $|\Xi_{24}'\rangle + |\Lambda\rangle$ is not anymore a projector if the $\{s_n\}$ sequence contains 0, since $|\Xi_0'(0)\rangle$ is not $*$–orthogonal to $|\Xi_{24}'\rangle$. This remark tells us that it not possible to have solutions representing configurations in which the string crosses the brane by a finite amount: the string has to stop at the brane.

This is to be contrasted with the configuration obtained by replacing $|\Lambda\rangle$ in $|\Xi_{24}'\rangle + |\Lambda\rangle$ with a D1–brane along the x axis, that is with $|\Xi_1'\rangle$. The state we get is definitely not a solution to the (4.6). This of course reinforces the interpretation of the $|\Lambda\rangle$ solution as a fundamental string.

Needless to say it is trivial to generalize the solution of the type $|\Xi_{24}'\rangle + |\Lambda\rangle$ to lower dimensional branes.

It is worth pointing out that it is also possible to construct string solutions of finite length. It is enough to choose the sequence $\{s_n\}$ to lie between two fixed values, say a and b in the x–axis, and then 'condense' the sequence between these two points. In the low energy limit the resulting solution shows precisely a flat profile for $a < x < b$ and a vanishing profile outside this interval (and of course a Gaussian profile along the other space direction). This solution is fit to represent a string stretched between two D–branes located at $x = a$ and $x = b$.

An important property for fundamental strings is the exchange property. Let us see if it holds for our solutions in a simple example. We consider first an extension of the solution (7.122) made of two pieces at right angles. Let us pick two space directions, x and y. We will denote by $\{s_n^x\}$ and $\{s_n^y\}$ a sequence of points along the positive x and y–axis. The string state

$$|\Lambda^{\pm\pm}\rangle = |\Xi_0'\rangle + \sum_{n=1}^{\infty} |\Xi_0'(\pm s_n^x)\rangle + \sum_{n=1}^{\infty} |\Xi_0'(\pm s_n^y)\rangle \tag{7.133}$$

is a solution to (4.6). The $\pm\pm$ label refers to the positive (negative) x and y–axis. This state represents an infinite string stretched along the positive (negative) x and y–axis including the origin. Now let us construct the string state

$$|\Xi_0'\rangle + \sum_{n=1}^{\infty} |\Xi_0'(s_n^x)\rangle + \sum_{n=1}^{\infty} |\Xi_0'(-s_n^x)\rangle + \sum_{n=1}^{\infty} |\Xi_0'(s_n^y)\rangle + \sum_{n=1}^{\infty} |\Xi_0'(-s_n^y)\rangle \tag{7.134}$$

This is still a solution to (4.6) and can be interpreted in two ways: either as $|\Lambda^{++}\rangle + |\Lambda^{--}\rangle$ or as $|\Lambda^{+-}\rangle + |\Lambda^{-+}\rangle$, up to addition to both of $|\Xi_0'\rangle$ (a bit removed from the origin). This addition costs the same amount of energy in the two cases, an amount that vanishes in the continuous limit. Therefore the solution (7.134) represents precisely the exchange property of fundamental strings.

So far we have considered only straight one–dimensional solutions (in terms of space profiles), or at most solutions represented by straight lines at right angles. However this is an unnecessary limitation. It is easy to generalize our construction to any curve in space. For instance, let us consider two directions in space and let us denote them again x and y ($\hat{p}^x$ and $\hat{p}^y$ being the relevant momentum operators). Let us construct the state

$$|\Xi_0'(s^x, s^y)\rangle = e^{-is^x\hat{p}^x} e^{-is^y\hat{p}^y} |\Xi_0'\rangle \tag{7.135}$$

It is evident that this represents a space–localized solution displaced from the origin by s^x in the positive x direction and s^y in the positive y direction. Using a suitable sequence $\{s_n^x\}$ and $\{s_n^y\}$, and rendering it dense, we can construct any curve in the $x-y$ plane, and, as a consequence, write down a solution to the equation of motion corresponding to this curve. The generalization to other space dimensions is straightforward.

We would like to remark that, by generalizing the above construction, one can also construct higher dimensional objects. For instance one could repeat the accretion construction by adding parallel D1–branes (that extend, say, in the y direction) along the x–axis. In this way we end up with a membrane-like configuration (with a flat profile in the x, y–plane), and continue in the same tune with higher dimensional configurations.

All the solutions we have considered so far are unstable. However the fundamental string solutions are endowed with a particular property. Since they end on a D–brane, their endpoints couple to the electromagnetic field on the brane, [102, 110, 111], and carry the corresponding charge. When the D–brane decays there is nothing that prevents the (fundamental) strings attached to it from decaying themselves. However in the presence of a background E–field, the latter are excited by the coupling with the E–field and persist (or, at least, persist longer than the other unstable objects).

Chapter 8

Conclusions

We begin this concluding chapter with a summary of our research.

Motivated by the search for a non perturbative definition of string theory, we have turned our attention to the phenomenon of Tachyon Condensation, which is a physical process which interpolates between a perturbative vacuum (on which the theory is initially quantized) and a non perturbative one (tachyon vacuum). Given the fact that the physical properties of such a vacuum are completely independent of the initial configuration of branes, it is very tempting to believe that Open String Field Theory formulated around it should manifest clear aspects of background independence.

This expectation has been put to test using the Vacuum String Field Theory model conjectured by Rastelli, Sen and Zwiebach. Although classical solutions representing any D–brane configuration can be easily obtained from star algebra projectors, they are all singular at the midpoint: this does not allow for a direct definition of observables, like their tension. We have indeed shown that the problem of finding classical solutions with finite energy density is equivalent, in the critical dimension $D = 26$, to the definition of the string coupling constant. This constant is not a free parameter of the purely ghost VSFT and (if we don't want to give up matter/ghost factorization) it can only emerge from a regularization procedure. We have provided and example of such regularization by introducing the dressing deformation: in particular the string coupling constant emerges from the tuning of the vanishing behaviour induced by the midpoint and the divergence induced by dressing. This is in general just a fine tuning (with the consequent lost of predictiveness) unless the theory is at the critical dimension $D = 26$. The critical dimension (which never enters in a naive approach to VSFT) emerges as a condition for the consistency of our regularization procedure.

The same dressing deformation is responsible for the implementation of the transversality condition of the $U(1)$ gauge field living on the $D25$–brane. All other massive modes can only be obtained from midpoint excitations of our classical solution, any other excitation is in fact trivial and can be gauged away.

The importance of the midpoint degree of freedom is further emphasized by the study of the string spectrum on multiple D–branes system. In particular, while the automatic generation of Chan Paton factors is encoded in the left/right degrees of freedom, the mass formula for strings stretched between parallel separated branes is the correct one only if one carefully regularize naively vanishing quantities coming from the midpoint. It should be stressed that, even if such anomalous quantities are the result of a breakdown of associativity, they nevertheless gives the correct observables in the VSFT framework.

As it emerges from our research, VSFT has proven to be extremely flexible even in the problem of finding time dependent backgrounds which describe the actual decay of a D–brane in real time. These solution are again projectors, but they drastically differ from the static solutions because they have an inverted eigenvalue in their Neumann matrix: it is just this simple modification that allows for localization in real time. Our procedure has proven to be general and works unambiguously even in a background E–field.

So far so good, is there something left aside? Even if the results obtained in this field are encouraging, there are many important unsolved problems which we think should be clearly understood. Here we list some of them.

- VSFT lacks of a complete regularization scheme. In this thesis many different regularizations have been used, many others are proposed in the literature: it seems that every problem needs a different regularization. This means that a complete regularization scheme has still to be found. It is possible that the correct regularization will spoil matter/ghost factorization, [113], which arises as a singular reparametrization of the string's worldsheet. The exact knowledge of the Tachyon Vacuum solution of OSFT would help in this direction.

- No one has been able, up to now, to concretely solve the OSFT equation of motion to get the tachyon vacuum solution. It is really not clear why a state which is so simply characterized in its physical properties should resist any attempt of analytic treatment. It is possible that the mathematical structure of OSFT is still too elementary to properly address this seemingly (and numerically) already–solved problem.

- The tachyon vacuum should be identified with a closed string vacuum, which one? A very interesting way of how closed strings amplitudes emerge from VSFT has been given in [22], however there is not evidence at all of how closed strings can emerge as asymptotic states: there is still much to understand in the non–perturbative implementation of Open/Closed duality. In particular there's no convincing understanding on how the shift in the closed string background created by a D–brane is encoded in the open string dynamics. Needless to say, this would enrich enormously our understanding of holography and of the gauge/gravity correspondence.

- What about superstrings? Although Sen's conjectures has been numerically tested in both cubic and WZW–like Open String Field Theory, there is no convincing formulation of Vacuum Superstring Field Theory. Why supersymmetry is so difficult to include in a string field theory framework? This can be a drawback of the RNS formulation (that can perhaps be overcome by better formulations like Berkovits' pure spinors) or there can be deeper problems that we still do not understand.

This is only a short list of some (in our opinion) important problems that should stimulate to work harder (but optimistically) in this subject.

Appendix A

Properties of Neumann coefficients

A.1 Proof of $U^2 = 1$

This section is devoted to a direct analytic proof of eqs.(2.29) and (2.79). Let us start from the latter.

Proof of eq.(2.79). It is convenient to rewrite it as follows

$$\sum_{k=0}^{\infty} \tilde{U}_{nk}\, \tilde{U}_{km} = \delta_{n0}\delta_{m0} + \sum_{k=0}^{\infty} \tilde{U}^{(2)}_{nk}\, \tilde{U}^{(1)}_{km} \tag{A.1}$$

since, in the range $0 \leq n, m < \infty$, we have $\tilde{U}^{(1)}_{km} = \delta_{n0}\delta_{m0} + \tilde{U}^{(2)}_{km}$ and $\tilde{U}^{(1)}_{0m} = \delta_{m0}$. Therefore we have to compute

$$\begin{aligned}\sum_{k=0}^{\infty} \tilde{U}^{(2)}_{nk}\, \tilde{U}^{(1)}_{km} \quad &= \quad \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{d\theta}{2\pi i} \oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}} \sum_{k=0}^{\infty} \frac{1}{(\zeta\theta)^{k+1}}\, \frac{f(z)}{f(\zeta)}\frac{f(\theta)}{f(w)} \cdot \\ &\cdot \quad \left(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta - z}\right)\left(\frac{1}{1+\theta w} - \frac{w}{w-\theta}\right)\end{aligned} \tag{A.2}$$

Here we have already exchanged the summation over k with integrals, which is allowed only under definite convergence conditions. The latter are guaranteed if $|\zeta\theta| > 1$, in which case

$$\sum_{k=0}^{\infty} \frac{1}{(\zeta\theta)^{k+1}} = \frac{1}{\theta\zeta - 1} \tag{A.3}$$

Now, we recall that, from the definition of $\tilde{U}^{(1)}, \tilde{U}^{(2)}$, we have $|z| < |\zeta|, |\theta| > |w|$. In order to comply with the condition $|\zeta\theta| > 1$ we choose to deform the θ contour while keeping the ζ contour fixed. In doing so we have to be careful to avoid possible singularities in θ. These are poles at $\theta = w, -\frac{1}{w}$ and branch cuts at $\theta = \pm i$, due to the $f(\theta)$ factor. One can deform the θ contour in such a way as to keep the pole at $-\frac{1}{w}$ external to the contour, since the w contour is as small as we wish around the origin. But, of course, one cannot avoid the branch points at $\theta = \pm i$. To make sense of the operation we introduce a regulator $K > 1$ and modify the integrand by modifying $f(\theta)$

$$f(\theta) \to f_K(\theta) = \left(\frac{K + i\theta}{K - i\theta}\right)^{\frac{2}{3}} \tag{A.4}$$

We will take K as large as needed and eventually move back to $K = 1$. Under these conditions we can safely perform the summation over k in (A.2) and make the replacement (A.3) in the integral.

As the next step we carry out the θ integration, which reduces to the contribution from the simple poles at $\theta = w$ and $\theta = \frac{1}{\zeta}$. The RHS of (A.2) becomes

$$= \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{d\zeta}{2\pi i} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \left[\frac{f(z)}{f(\zeta)} \frac{f_K(1/\zeta)}{f(w)} \left(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta - z} \right) \left(\frac{1}{w+\zeta} - \frac{w}{\zeta w - 1} \right) \right.$$
$$\left. + \frac{f(z)}{f(\zeta)} \frac{w}{\zeta w - 1} \left(\frac{1}{1+z\zeta} - \frac{\zeta}{\zeta - z} \right) \right] \tag{A.5}$$

The first line corresponds to the contribution from the pole at $\theta = \frac{1}{\zeta}$, while the second comes from the pole at $\theta = w$.

Next we wish to integrate with respect to ζ. The singularities trapped within the ζ contour of integration are the poles at $\zeta = z, -w$ (not the poles at $\zeta = \frac{1}{w}, -\frac{1}{z}$). Since above we had $K > |\theta| > \frac{1}{|\zeta|}$, it follows that $|\zeta| > \frac{1}{K}$. Therefore also the branch points at $\zeta = \pm\frac{i}{K}$ of $f_K(1/\zeta)$ are trapped inside the ζ contour and we have to compute the relevant contribution to the integral. In the integrand of (A.5) we have two cuts in ζ. One is the cut we have just mentioned, let us call it $\mathfrak{c}_{1/K}$ and let us fix it to be the semicircle of radius $1/K$ at the RHS of the imaginary axis; the contour that surrounds it excluding all the other singularities will be denoted $C_{1/K}$. The other cut, due to $f(\zeta)$, with branch points at $\zeta = \pm i$, will be denoted $\mathfrak{c}_1$; the contour that surrounds it excluding all the other singularities will be denoted C_1.

After these lengthy preliminaries let us carry out the integration over ζ. We get

$$= \oint \frac{dz}{2\pi i} \frac{1}{z^{n+1}} \oint \frac{dw}{2\pi i} \frac{1}{w^{m+1}} \left[\frac{f(1/z)}{f(w)} \left(\frac{zw}{zw-1} - \frac{z}{z+w} \right) + \frac{f(z)}{f(1/w)} \left(\frac{1}{1-zw} - \frac{w}{w+z} \right) \right.$$
$$\left. + \oint_{C_{1/K}} \frac{d\zeta}{2\pi i} (\dots) + \frac{zw}{1-zw} \right] \tag{A.6}$$

The first two terms in square brackets come from the contribution of the poles at $\zeta = z$ and $\zeta = -w$ from the first line in (A.5), respectively. The symbol $(\dots)$ represents the integrand contained within the square brackets in the first line of (A.5). Finally the last term in (A.6) is the contribution coming from the second line of (A.5) due to the pole at $\zeta = z$. We notice that

$$\frac{zw}{zw-1} - \frac{z}{z+w} = \frac{w}{z+w} - \frac{1}{1-zw} \tag{A.7}$$

but of course the problem here is how to evaluate the integral around the cut. Fortunately this can be reduced to an evaluation of contributions from poles. To see this, we first recall the properties of $f(z)$. It is easy to see that

$$f(1/z) = f(-z) \quad \text{and} \quad f(-z) = 1/f(z) \tag{A.8}$$

Therefore, in the limit $K \to 1$, the factor $f_K(1/\zeta)/f(\zeta)$ tends to $(f(-\zeta))^2$. As a consequence, in the same limit, the integral of $(\ldots)$ around the cut $\mathfrak{c}_{1/K}$ is the same as the integral around the cut $\mathfrak{c}_1$, and each equals one–half the integral around both contours, in other words each equals one–half the integral about a contour that surrounds both cuts and exclude all the other singularities (which are poles). By a well-known argument, the latter integral equals the negative of the integral of $(\ldots)$ about all the remaining singularities in the complex ζ–plane. This is easy to compute. The remaining singularities are poles around $\zeta = z, -w, -1/z, 1/w$. Notice that there is no singularity at $\zeta = \infty$. Carrying out this calculation explicitly we get

$$\begin{aligned} = & \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}}\oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}}\left\{ \frac{f(1/z)}{f(w)}\left(\frac{w}{z+w}-\frac{1}{1-zw}\right)+\frac{f(z)}{f(1/w)}\left(\frac{1}{1-zw}-\frac{w}{w+z}\right)\right. \\ & -\frac{1}{2}\left[\frac{f(1/z)}{f(w)}\left(\frac{w}{z+w}-\frac{1}{1-zw}\right)+\frac{f(z)}{f(1/w)}\left(\frac{1}{1-zw}-\frac{w}{w+z}\right)\right. \\ & \left.\left.+\frac{f(1/z)}{f(w)}\left(\frac{w}{z+w}-\frac{1}{1-zw}\right)+\frac{f(z)}{f(1/w)}\left(\frac{1}{1-zw}-\frac{w}{w+z}\right)\right]+\frac{zw}{1-zw}\right\} \end{aligned} \tag{A.9}$$

The terms in square brackets represent the contribution from the cut $\mathfrak{c}_{1/K}$ and come from the simple poles at $\zeta = z, -w, -1/z, 1/w$, respectively. All the terms cancel out except the last in the third line. So the RHS of (A.2) reduces to

$$= \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}}\oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}}\sum_{k=1}^{\infty}(zw)^k = \delta_{nm}, \qquad n,m \geq 1 \tag{A.10}$$

This complete the proof of (2.29). We remark that we could have integrated first with respect to ζ and then with respect to θ. The procedure is somewhat different, but the final result is the same. We also point out that there may be other equivalent ways to derive (2.29).

Proof of eq.(2.29). It is convenient to rewrite U_{nm} in an alternative form compared to (2.24). We start by replacing in eq.(2.20)

$$f_a'(z)\frac{1}{(f_a(z)-f_b(w))^2}f_b'(w) = -\partial_z \frac{1}{f_a(z)-f_b(w)}f_b'(w) \tag{A.11}$$

and integrating by part. We decompose the resulting expression as in eq.(2.22). After some algebra one gets

$$U_{nm} = \sqrt{\frac{n}{m}}\oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}}\oint \frac{dw}{2\pi i}\frac{1}{w^{m+1}}\frac{g(z)}{g(w)}\left(\frac{1}{1+zw}-\frac{w}{w-z}\right) \tag{A.12}$$

where

$$g(z) = \frac{1}{z}(1+iz)^{\frac{2}{3}}(1-iz)^{\frac{4}{3}} \tag{A.13}$$

This function satisfies

$$g(1/z) = g(-z) \tag{A.14}$$

which corresponds to the first of eqs.(A.8). There is no analog of the second.

In order to prove eq.(2.29) we have to evaluate

$$\sqrt{\frac{m}{n}}\sum_{k=1}^{\infty} U_{nk}U_{km} = \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}}\oint\frac{d\zeta}{2\pi i}\oint\frac{d\theta}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{w^{m+1}}\sum_{k=1}^{\infty}\frac{1}{(\zeta\theta)^{k+1}}\frac{g(z)}{g(\zeta)}\frac{g(\theta)}{g(w)}\cdot$$
$$\cdot\left(\frac{1}{1+z\zeta}-\frac{\zeta}{\zeta-z}\right)\left(\frac{1}{1+\theta w}-\frac{w}{w-\theta}\right) \qquad \text{(A.15)}$$

The structure is the same as in (A.2), except for the substitution $f \to g$ and for the fact that now the summation over k starts from 1. We will thus proceed as above while paying attention to the differences. Using

$$\sum_{k=1}^{\infty}\frac{1}{(\zeta\theta)^{k+1}} = \frac{1}{\zeta\theta}\frac{1}{\theta\zeta-1} \qquad \text{(A.16)}$$

instead of (A.3), we see that, when integrating over θ we have to take into account the pole at $\theta = 0$. The result is

$$= \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}}\oint\frac{d\zeta}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{w^{m+1}}\left[\frac{g(z)}{g(\zeta)}\frac{g_K(1/\zeta)}{g(w)}\left(\frac{1}{1+z\zeta}-\frac{\zeta}{\zeta-z}\right)\left(\frac{1}{w+\zeta}-\frac{w}{\zeta w-1}\right)\right.$$
$$\left.+\frac{f(z)}{f(\zeta)}\frac{w}{\zeta w-1}\left(\frac{1}{1+z\zeta}-\frac{\zeta}{\zeta-z}\right)+\frac{g(z)}{\zeta g(\zeta)g(w)}\left(\frac{1}{1+z\zeta}-\frac{\zeta}{\zeta-z}\right)\frac{1+w^2}{w}\right] \qquad \text{(A.17)}$$

The last contribution comes precisely from the double pole at $\theta = 0$.

Next let us integrate over ζ. There is no singularity at $\zeta = 0$ or $\zeta = \infty$, as one may have suspected. Let us deal first with the first line in eq.(A.17). This is exactly the first line of (A.5), except for the substitution $f \to g$. We proceed in the same way as above, but with some additional care because we cannot use the analog of the second eq.(A.8). However we remark that

$$\frac{g_K(1/\zeta)}{g(\zeta)} = \frac{f_K(1/\zeta)}{f(\zeta)}\frac{(\zeta K-i)^2}{(1-i\zeta)^2} \qquad \text{(A.18)}$$

Now we have recovered the same structure as in (A.5) except for the last factor in the RHS of (A.18), i.e. at the price of bringing into the game a double pole at $\zeta = -i$. Fortunately the residue of this pole vanishes. All is well what ends well. We can now safely repeat the same argument that leads from eq.(A.5) to eq.(A.9), and conclude that the various contributions from the first line of eq.(A.17) add up to zero. The second line is easy to compute, the only contribution comes from the simple pole at $\zeta = z$:

$$= \oint \frac{dz}{2\pi i}\frac{1}{z^{n+1}}\oint\frac{d\zeta}{2\pi i}\oint\frac{dw}{2\pi i}\frac{1}{w^{m+1}}\left[\frac{1}{1-zw}-\frac{1}{g(w)}\frac{1+w^2}{w}\right] = \delta_{nm}, \qquad n,m \geq 1 \quad \text{(A.19)}$$

This completes the proof of (2.29).

A.2 A collection of well–known formulae

In this Appendix we collect some useful results and formulas involving the matrices of the three strings vertex coefficients.

To start with, we recall that

- (i) V^{rs}_{nm} are symmetric under simultaneous exchange of the two couples of indices;
- (ii) they are endowed with the property of cyclicity in the r, s indices, i.e. $V^{rs} = V^{r+1,s+1}$, where $r, s = 4$ is identified with $r, s = 1$.

Next, using the twist matrix C ($C_{mn} = (-1)^m \delta_{mn}$), we define

$$X^{rs} \equiv CV^{rs}, \qquad r, s = 1, 2, \tag{A.20}$$

These matrices are often rewritten in the following way $X^{11} = X$, $X^{12} = X_+$, $X^{21} = X_-$. They commute with one another

$$[X^{rs}, X^{r's'}] = 0, \tag{A.21}$$

moreover

$$CV^{rs} = V^{sr}C, \qquad CX^{rs} = X^{sr}C \tag{A.22}$$

Next we quote some useful identities:

$$\begin{aligned} &X + X_+ + X_- = 1 \\ &X_+X_- = (X)^2 - X \\ &(X_+)^2 + (X_-)^2 = 1 - (X)^2 \\ &(X_+)^3 + (X_-)^3 = 2(X)^3 - 3(X)^2 + 1 \end{aligned} \tag{A.23}$$

and

$$\frac{1 - TX}{1 - X} = \frac{1}{1 - T}, \qquad \frac{X}{1 - X} = \frac{T}{(1 - T)^2} \tag{A.24}$$

Using these one can show, for instance, that

$$\begin{aligned} \mathcal{K}^{-1} &= \frac{1}{(1 + T)(1 - X)} \begin{pmatrix} 1 - TX & TX_+ \\ TX_- & 1 - TX \end{pmatrix} \\ \mathcal{M}\mathcal{K}^{-1} &= \frac{1}{(1 + T)(1 - X)} \begin{pmatrix} (1 - TX)X & X_+ \\ X_- & (1 - TX)X \end{pmatrix} \end{aligned} \tag{A.25}$$

Another ingredient we need is given by the Fock space projectors

$$\rho_1 = \frac{1}{(1 + T)(1 - X)} \left[X_+(1 - TX) + T(X_-)^2\right] \tag{A.26}$$

$$\rho_2 = \frac{1}{(1 + T)(1 - X)} \left[X_-(1 - TX) + T(X_+)^2\right] \tag{A.27}$$

They satisfy

$$\rho_1^2 = \rho_1, \qquad \rho_2^2 = \rho_2, \qquad \rho_1 + \rho_2 = 1, \qquad \rho_1\rho_2 = 0 \tag{A.28}$$

i.e. they project onto orthogonal subspaces. Moreover,

$$\rho_1^T = \rho_1 = C\rho_2 C, \qquad \rho_2^T = \rho_2 = C\rho_1 C. \tag{A.29}$$

where ^{T} represents matrix transposition. As was shown in [30, 34], ρ_1, ρ_2 projects out half the string modes. Using these projectors one can prove that

$$(X_+, X_-)\,\mathcal{K}^{-1} = (\rho_1, \rho_2), \qquad \mathcal{M}\mathcal{K}^{-1}\mathcal{T}\begin{pmatrix} X_- \\ X_+ \end{pmatrix} = \begin{pmatrix} TX\rho_2 + TX_+\rho_1 \\ TX_-\rho_2 + TX\rho_1 \end{pmatrix} \tag{A.30}$$

which are used throughout the paper.

The following relations are often useful

$$\rho_1 X_+ + \rho_2 X_- = 1 - XT, \qquad \rho_1 X_- + \rho_2 X_+ = X(T-1) \tag{A.31}$$

The next set of equations involve $\mathbf{v}_0, \mathbf{v}_\pm$. We start with

$$\begin{aligned} &\mathbf{v}_+ + \mathbf{v}_- + \mathbf{v}_0 = 0 \\ &\mathbf{v}_0^2 + \mathbf{v}_+^2 + \mathbf{v}_-^2 = \frac{4}{3}V_{00} \\ &\mathbf{v}_0\mathbf{v}_- + \mathbf{v}_0\mathbf{v}_+ + \mathbf{v}_-\mathbf{v}_+ = -\frac{2}{3}V_{00} \end{aligned} \tag{A.32}$$

Next we have the representation in terms of $\mathbf{v}_0$

$$\begin{aligned} \mathbf{v}_+ &= \frac{1}{1+T}\left[(T-2)\rho_2 + (1-2T)\rho_1\right]\mathbf{v}_0 \\ \mathbf{v}_- &= \frac{1}{1+T}\left[(T-2)\rho_1 + (1-2T)\rho_2\right]\mathbf{v}_0 \end{aligned}$$

from which we get

$$\begin{aligned} \mathbf{v}_+ - \mathbf{v}_0 &= -\frac{3}{1+T}(\rho_2 + T\rho_1)\mathbf{v}_0 \\ \mathbf{v}_+ - \mathbf{v}_- &= -3\frac{1-T}{1+T}(\rho_2 - \rho_1)\mathbf{v}_0 \\ \mathbf{v}_- - \mathbf{v}_0 &= -\frac{3}{1+T}(\rho_1 + T\rho_2)\mathbf{v}_0 \end{aligned} \tag{A.33}$$

Using these equations in (A.32) it is easy to obtain in particular

$$\frac{2}{3}V_{00} = 3\langle\mathbf{v}_0|\frac{T^2 - T + 1}{(1+T)^2}|\mathbf{v}_0\rangle = \langle\mathbf{t}_0|\frac{1}{1+T}|\mathbf{v}_0\rangle \tag{A.34}$$

where $\mathbf{t}_0 = 3\frac{T^2-T+1}{T+1}|\mathbf{v}_0\rangle$.

We often use the continuous basis to evaluate various brackets which appear in the computations. We therefore need the matrices and vectors that define the 3 strings vertex in the k–basis. We use normalized k-vectors, see [80],

$$|k\rangle = \sum_{n=1}^{\infty}\frac{1}{k}\sqrt{\frac{nk}{2\sinh\frac{\pi k}{2}}}\oint\frac{dz}{2\pi i}\frac{1}{z^{n+1}}\left(1 - \exp(-k\tan^{-1}z)\right)|n\rangle$$

so that

$$\langle k|k'\rangle = \delta(k-k')$$

With these conventions we have

$$\begin{aligned}
X &= \int_{-\infty}^{\infty} dk X(k)|k\rangle\langle k|, \qquad & X(k) &= -\frac{1}{1+2\cosh\frac{\pi k}{2}}\\
T &= \int_{-\infty}^{\infty} dk T(k)|k\rangle\langle k|, \qquad & T(k) &= -\mathrm{e}^{-\frac{\pi|k|}{2}}\\
\rho_1 &= \int_0^{\infty} dk\,|k\rangle\langle k|, \qquad & \rho_2 &= \int_{-\infty}^{0} dk\,|k\rangle\langle k|
\end{aligned} \tag{A.35}$$

and

$$\begin{aligned}
|\mathbf{v}_0\rangle &= \int_{-\infty}^{\infty} dk v_0(k)|k\rangle \qquad & v_0(k) &= -\frac{4}{3k}\sqrt{\frac{k}{\sinh\frac{\pi k}{2}}}\frac{\sinh^2\frac{\pi k}{4}}{1+2\cosh\frac{\pi k}{2}}\\
|\mathbf{t}_0\rangle &= \int_{-\infty}^{\infty} dk t_0(k)|k\rangle \qquad & t_0(k) &= -\frac{4}{k\left(e^{\frac{\pi|k|}{2}}-1\right)}\sqrt{\frac{k}{\sinh\frac{\pi k}{2}}}\sinh^2\frac{\pi k}{4}
\end{aligned} \tag{A.36}$$

All other matrices and vectors can be easily obtained using the properties (A.31) and (A.33). Notice that, since $C|k\rangle = -|-k\rangle$, twist even vectors are represented by odd functions and viceversa.

Notice also that $\mathbf{t}_0$ has a jump discontinuity in $k=0$

$$t_0(0^+) = -t_0(0^-) = -\sqrt{\frac{\pi}{2}}$$

In the ghost sector matrices $\widetilde{X}$, $\widetilde{X}_\pm$ satisfy all relations above. In particular one can define the half–string projectors $\tilde\rho_1, \tilde\rho_2$ as in (A.26,A.27) and verify that they satisfy the same relations as the matter projectors. For the manipulations with zero-modes it is useful here to define the vectors

$$(\widetilde{\mathbf{v}}_0)_n = \widetilde{V}^{r,r}_{n0}\ , \qquad (\widetilde{\mathbf{v}}_\pm)_n = \widetilde{V}^{r,r\pm1}_{n0} \tag{A.37}$$

which satisfy

$$\widetilde{\mathbf{v}}_0 = (1-\widetilde{X})\mathbf{f}\ , \qquad \widetilde{\mathbf{v}}_\pm = -\widetilde{X}_\mp\mathbf{f} \tag{A.38}$$

where $\mathbf{f} = \{f_n\}$ is given by

$$f_n = \cos\left(\frac{n\pi}{2}\right)\ . \tag{A.39}$$

This vector $\mathbf{f}$ appears in the expression for the kinetic operator $\mathcal{Q}$:

$$\mathcal{Q} = c_0 + \sum_{n=1}^{\infty} f_n\left(c_n + (-1)^n c_n^\dagger\right)\ . \tag{A.40}$$

Appendix B

Diagonal representation of CU'

With reference to formula (2.119), we illustrate the spectroscopy and diagonal representation of CU'. The matrix CU' is hermitian, unitary and commutes with $U'C$. The discrete eigenvalues ξ and $\bar{\xi}$ are determined as follows, [50]. Let

$$\xi = -\frac{2 - \cosh\eta - i\sqrt{3}\sinh\eta}{1 - 2\cosh\eta} \tag{B.1}$$

and

$$F(\eta) = \psi\left(\frac{1}{2} + \frac{\eta}{2\pi i}\right) - \psi\left(\frac{1}{2}\right), \qquad \psi(z) = \frac{d\log\Gamma(z)}{dz} \tag{B.2}$$

Then the eigenvalues ξ and $\bar{\xi}$ are the solutions of

$$\Re F(\eta) = \frac{b}{4} \tag{B.3}$$

The eigenvectors $V_n^{(\xi)}$ are defined via the generating function

$$\begin{aligned} F^{(\xi)}(z) = \quad & \sum_{n=1}^{\infty} V_n^{(\xi)} \frac{z^n}{\sqrt{n}} = -\sqrt{\frac{2}{b}} V_0^{(\xi)} \left[\frac{b}{4} + \frac{\pi}{2\sqrt{3}}\frac{\xi - 1}{\xi + 1} + \log iz \right. \\ & \left. + e^{-2i(1+\frac{\eta}{\pi i})\arctan z}\Phi(e^{-4i\arctan z}, 1, \frac{1}{2} + \frac{\eta}{2\pi i})\right] \end{aligned} \tag{B.4}$$

where $\Phi(x, 1, y) = 1/y\,{}_2F_1(1, y; y + 1; x)$, while

$$V_0^{(\xi)} = \left(\sinh\eta \frac{\partial}{\partial\eta}\left[\log\Re F(\eta)\right]\right)^{-\frac{1}{2}} \tag{B.5}$$

As for the continuous spectrum, it is spanned by the variable k, $-\infty < k < \infty$. The eigenvalues of CU' are given by

$$\nu(k) = -\frac{2 + \cosh\frac{\pi k}{2} + i\sqrt{3}\sinh\frac{\pi k}{2}}{1 + 2\cosh\frac{\pi k}{2}}$$

The generating function for the eigenvectors is

$$F_c^{(k)}(z) = \sum_{n=1}^{\infty} V_n^{(k)} \frac{z^n}{\sqrt{n}} = V_0^{(k)} \sqrt{\frac{2}{b}} \left[-\frac{b}{4} - \left(\Re F_c(k) - \frac{b}{4} \right) e^{-k \arctan z} - \log iz \right. \tag{B.6}$$

$$\left. - \left(\frac{\pi}{2\sqrt{3}} \frac{\nu(k)-1}{\nu(k)+1} + \frac{2i}{k} \right) + 2i\, f^{(k)}(z) - \Phi(e^{-4i \arctan z}, 1, 1+\frac{k}{4i})\, e^{-4i \arctan z}\, e^{-k \arctan z} \right]$$

where

$$F_c(k) = \psi(1+\frac{k}{4\pi i}) - \psi(\frac{1}{2}),$$

while

$$V_0^{(k)} = \sqrt{\frac{b}{2\mathcal{N}(k)}} \left[4 + k^2 \left(\Re F_c(k) - \frac{b}{4} \right)^2 \right]^{-\frac{1}{2}} \tag{B.7}$$

The continuous eigenvalues of X', X'_-, X'_- and T' (for the conventional lump) are given by same formulas as for the X, X_+, X_- and T case, eqs(2.115,2.116). As for the discrete eigenvalues, they are given by the formulas

$$\begin{aligned} \mu_\xi^{rs} &= \frac{1 - 2\,\delta_{r,s} - e^{\eta}\,\delta_{r+1,s} - e^{-\eta}\,\delta_{r,s+1}}{1 - 2\cosh \eta} \\ t_\xi &= e^{-|\eta|} \end{aligned} \tag{B.8}$$

B.1 Limits of X' and T'

In this Appendix we briefly discuss the low energy and high energy limit of X' and T' in the oscillator basis. The Neumann coefficients $V_{NM}^{\prime(rs)}$ we use are given in Appendix B of [54]. They explicitly depend on the b parameter. In the low energy limit the three–strings vertex can be expanded by means of a parameter ϵ (a dimensionless parameter, in fact an alias of α') , see [64]. This translates into an expansion for $V_{NM}^{\prime(rs)}$ triggered by the following rescalings

$$\begin{aligned} V_{mn}^{(rs)} &\to V_{mn}^{(rs)} \\ V_{m0}^{(rs)} &\to \sqrt{\epsilon} V_{m0}^{(rs)} \\ V_{00} &\to \epsilon V_{00} \end{aligned} \tag{B.9}$$

For instance X' is expanded as follows to the lowest orders of approximation

$$X' = \begin{pmatrix} -\frac{1}{3} + \frac{8}{3} V_{00} \frac{\epsilon}{b} & -\frac{4}{3}\sqrt{\frac{2\epsilon}{b}} \langle \mathbf{v}_e | \\ -\frac{4}{3}\sqrt{\frac{2\epsilon}{b}} | \mathbf{v}_e \rangle & X - \frac{8}{3}\frac{\epsilon}{b} (|\mathbf{v}_e\rangle\langle \mathbf{v}_e| - |\mathbf{v}_o\rangle\langle \mathbf{v}_o|) \end{pmatrix} \tag{B.10}$$

where

$$|\mathbf{v}_e\rangle_n = -\frac{3}{2\sqrt{2}} V_{0n}^{(11)}, \qquad |\mathbf{v}_o\rangle_n = \sqrt{\frac{3}{8}} (V_{0n}^{(12)} - V_{0n}^{(21)})$$

It is interesting to remark that the parameter ϵ appears always divided by b, so that one could just as well absorb ϵ into $1/b$ and say that the expansion is in the parameter $1/b$ for large b. However to avoid confusion it is useful to keep the two parameters distinct.

Now, it is immediate to see that

$$T' = \begin{pmatrix} -1+\mathcal{O}(\frac{\epsilon}{b}) & \mathcal{O}(\sqrt{\frac{\epsilon}{b}}) \\ \mathcal{O}(\sqrt{\frac{\epsilon}{b}}) & T+\mathcal{O}(\frac{\epsilon}{b}) \end{pmatrix} \tag{B.11}$$

This is correct provided we can prove that the use of (B.10) to compute T' makes full sense, that is all the terms of the expansion in powers of $\sqrt{\frac{\epsilon}{b}}$ are well defined. One can actually see that a naive expansion leads to infinite coefficients. This is a well–known problem, pointed out for the first time in [64], which requires a regularization. A nice way to introduce a regulator is to switch on a constant background B field. We will not do it here, but we quote the result: in the presence of a B field the infinities disappear, and the expansion (B.11) makes full sense. From this we deduce in particular that

$$T'_{nm} = T_{nm} + \mathcal{O}(\frac{\epsilon}{b}) \tag{B.12}$$

This result is used in Section 6.

Let us consider now another extreme expansion, that is the limit $\alpha' \to \infty$. In just the same way as above, we can introduce an alias, t ($t >> 1$) instead of ϵ. So, in particular,

$$\begin{aligned} &V^{(rs)}_{mn} \to V^{(rs)}_{mn} \\ &V^{(rs)}_{m0} \to \sqrt{t}V^{(rs)}_{m0} \\ &V_{00} \to tV_{00} \end{aligned} \tag{B.13}$$

In this case X' to the lowest orders of approximation becomes

$$X' = \begin{pmatrix} 1+\frac{2}{3}\frac{1}{V_{00}}\frac{b}{t} & -\frac{2}{3}\sqrt{\frac{2b}{t}}\langle \mathbf{v}_e| \\ -\frac{2}{3}\sqrt{\frac{2b}{t}}|\mathbf{v}_e\rangle & X-\frac{4}{3}\frac{1}{V_{00}}(1-\frac{1}{V_{00}}\frac{b}{2t})(|\mathbf{v}_e\rangle\langle\mathbf{v}_e| - |\mathbf{v}_o\rangle\langle\mathbf{v}_o|) \end{pmatrix} \tag{B.14}$$

The lowest order in this expansion is known as the tensionless limit [98]. Also here one must be careful about the use of this expansion in calculating T'. From eq.(B.14) one finds that

$$T'_{00} = 1 + \mathcal{O}(\frac{b}{t}) \tag{B.15}$$

B.2 The $\alpha' \to 0$ limit of $\check{S}'^{(c)}_{nm}$ and $\check{S}'^{(c)}_{0n}$

In this Appendix we discuss the limit of the unconventional lump matrix elements $\check{S}'^{(c)}_{nm}$ and $\check{S}'^{(c)}_{0n}$ by means of the diagonal basis. According to (7.11), we speak interchangeably of the $b \to \infty$ limit and the $\eta \to \infty$ one. When applying the results of this Appendix to section 6, we understand that $1/b$ is replaced everywhere by ϵ/b with finite b.

As a preliminary step let us prove that

$$\lim_{b\to\infty}\left(V_0^{(k)}\right)^2 = \delta(k) \tag{B.16}$$

A rather informal way to see this is as follows. Looking at (B.7) it is easy to realize that the limit always vanishes provided $k \neq 0$. Therefore the support of the limiting distribution must be at $k = 0$. We can therefore expand all the functions involved in k around $k = 0$ and keep the leading terms. Since $\Re F_c(k) \approx 1.386...$ around this point, we can disregard $\Re F_c(k)$ compared to $b/4$ in the $b \to \infty$ limit. Therefore we easily find

$$\lim_{b\to\infty}(V_0^{(k)})^2 = \lim_{b\to\infty} = \frac{\bar{b}}{\pi}\frac{1}{1+\bar{b}^2k^2}$$

where $\bar{b} = b/8$. Now defining $\bar{\epsilon} = 1/\bar{b}$, the limit becomes

$$\lim_{\bar{\epsilon}\to 0}\frac{1}{\pi}\frac{\bar{\epsilon}}{k^2+\bar{\epsilon}^2} = \delta(k) \tag{B.17}$$

according to a well–known representation of the delta function. We can also show that

$$(V_0^{(k)})^2 = \delta(k) + \mathcal{O}(1/b)$$

From now on we suppose that, in the $\int dk$ integrals , we are allowed to replace the integrands with their $1/b$ expansions, and that the results we obtain are valid at least in an asymptotic sense. This attitude is always confirmed by numerical approximations.

B.2.1 Limit of $\check{S}'^{(c)}_{mn}$

Let us rewrite the generating function for $V_m^{(k)}$ as follows:

$$F^{(k)}(z) = A^{(k)}f^{(k)}(z) - \frac{(1-\nu(k))V_0^{(k)}}{\sqrt{b}}B(k,z) \tag{B.18}$$

where

$$A^{(k)} = V_0^{(k)}\sqrt{\frac{2}{b}}k\left(\Re F_c(k) - \frac{b}{4}\right) \tag{B.19}$$

and

$$\begin{aligned} B(k,z) &= \frac{2}{1-\nu(k)}\Bigg[\Re F_c(k) + \frac{\pi}{2\sqrt{3}}\frac{\nu(k)-1}{\nu(k)+1} + \frac{2i}{k} + \log(iz) - 2if^{(k)}(z) \\ &+ \mathrm{LerchPhi}(e^{-4i\mathrm{arctan}(z)}, 1, 1+\frac{k}{4i})e^{-4i\mathrm{arctan}(z)}e^{-k\mathrm{arctan}(z)}\Bigg] \end{aligned} \tag{B.20}$$

From (B.18) we can derive a useful expression for $V_m^{(k)}$:

$$V_m^{(k)} = A^{(k)}\frac{\sqrt{m}}{2\pi i}\oint dz\frac{f^{(k)}(z)}{z^{m+1}} - \frac{(1-\nu(k))V_0^{(k)}}{\sqrt{b}}\frac{\sqrt{m}}{2\pi i}\oint dz\frac{B(k,z)}{z^{m+1}} \tag{B.21}$$

Since $v_m^{(k)} = \frac{\sqrt{m}}{2\pi i}\oint dz \frac{f^{(k)}(z)}{z^{m+1}}$ and $S'^{(c)}_{mn} = \int_{-\infty}^{\infty} dk\ t(k) V_m^{(k)} V_n^{(-k)}$ we get:

$$\begin{aligned}\check{S}'^{(c)}_{mn} &= \int_{-\infty}^{\infty} dk\ t(k)\Big[A^{(k)}A^{(-k)} v_m^{(k)} v_n^{(-k)} - A^{(k)} V_0^{(k)} v_m^{(k)} (1-\bar{\nu}(k))\tilde{B}_n(-k)\frac{1}{\sqrt{b}} \\ &- A^{(-k)} V_0^{(k)} v_n^{(-k)}(1-\nu(k))\tilde{B}_m(k)\frac{1}{\sqrt{b}} + (V_0^{(k)})^2 (1-\bar{\nu}(k))(1-\nu(k))\tilde{B}_m(k)\tilde{B}_n(-k)\frac{1}{b}\Big]\end{aligned} \tag{B.22}$$

where

$$\tilde{B}_m(k) = \frac{\sqrt{m}}{2\pi i}\oint dz \frac{B(k,z)}{z^{m+1}}$$

Now we want to take the limit of (B.22) when $b \to \infty$. To this end we notice the following:

$$\begin{aligned}\lim_{b\to\infty} A^{(k)}A^{(-k)} &= \lim_{b\to\infty} (V_0^{(k)})^2 \left(\frac{-2k^2}{b}\right)\left(\Re F_c(k) - \frac{b}{4}\right)^2 \\ &= \lim_{x\to-\infty} \left(\frac{-k^2}{N(k)}\right)\frac{x^2}{4+k^2x^2} = \left(\frac{-k^2}{N(k)}\right)\frac{1}{k^2} = -\frac{1}{N(k)}\end{aligned}$$

where $x = \left(\Re F_c(k) - \frac{b}{4}\right)$. When k is very large $\Re F_c(k)$ tends to (slowly) diverge, but the factor $t(k)$ in the integrand of (B.22) concentrates the integral in the small k region.

We also need:

$$\begin{aligned}\lim_{b\to\infty} \frac{A^{(k)}V_0^{(k)}}{\sqrt{b}} &= \sqrt{2}k\delta(k)\left(\frac{\Re F_c(k)}{b} - \frac{1}{4}\right) \\ \lim_{b\to\infty} \frac{A^{(-k)}V_0^{(k)}}{\sqrt{b}} &= -\sqrt{2}k\delta(k)\left(\frac{\Re F_c(k)}{b} - \frac{1}{4}\right)\end{aligned}$$

Finally using these limits

$$\begin{aligned}\lim_{b\to\infty} \check{S}'^{(c)}_{mn} &= -\int_{-\infty}^{\infty} \frac{dk}{N(k)}\ t(k) v_m^{(k)} v_n^{(-k)} \\ &+ \lim_{b\to\infty}\int_{-\infty}^{\infty} dk\ t(k)\delta(k)(1-\bar{\nu}(k))(1-\nu(k))\tilde{B}_m(k)\tilde{B}_n(-k)\frac{1}{b}\end{aligned}$$

while the other integrals vanish because they contain the factor $k\delta(k)$. Here we have used the fact that $\nu(0) = \bar{\nu}(0) = -1$ and $\tilde{B}_m(0)$ is finite, for a straightforward calculation gives

$$\tilde{B}_m(0) = \frac{\sqrt{m}}{2\pi i}\oint dz \frac{\log(1+z^2)}{z^{m+1}} = \begin{cases} 0 & \text{for m odd;} \\ \frac{\sqrt{2m}}{2}(-1)^{\frac{m}{2}+1}(\frac{m}{2}+1)! & \text{for m even.}\end{cases} \tag{B.23}$$

So we are left with:

$$\lim_{b\to\infty} \check{S}'^{(c)}_{mn} = S_{mn} \tag{B.24}$$

This is the sliver. The corrections are of order $\frac{1}{b}$.

B.2.2 Limit of $\check{S}'_{0m}$

In the rest of this appendix we would like to justify eq.(7.35). The limit of $\check{S}'^{(c)}_{0m}$ can be computed the same way as before. We have:

$$\lim_{b\to\infty} \check{S}'^{(c)}_{0m} = \lim_{b\to\infty} \int_{-\infty}^{\infty} dk\; t(k) V_0^{(k)} V_m^{(-k)} = \lim_{b\to\infty} \left(\int_{-\infty}^{\infty} dk\; t(k) V_0^{(k)} A^{(-k)} v_m^{(-k)} + \frac{2}{\sqrt{b}} \tilde{B}_m(0) \right) \tag{B.25}$$

The last term in the RHS of course vanishes in the limit $b \to \infty$, while the first limit diverges, but, recalling (7.35), what we are really need to know is the limit of $\frac{\check{S}'_{0n}}{1+s'}$. Using the fact that $1 + s' \approx 4\eta \log \eta$ when $b \to \infty$ ($b \approx 4\log \eta$) and that we can write $\check{S}'_{0m} = \check{S}'^{(c)}_{0m} + \check{S}'^{(d)}_{0m}$ (factorization into continuous and discrete parts) we have:

$$\begin{aligned} \check{S}'^{(d)}_{0m} &\approx 2\eta\sqrt{2\log\, \eta} \\ \check{S}'^{(c)}_{0m} &\approx \int_{-\infty}^{\infty} dk\; t(k) v_m^{(-k)} (-\sqrt{2}k)(V_0^{(k)})^2 \left(\frac{\Re F_c(k)}{4\log\, \eta} - \frac{1}{4} \right) 2\sqrt{\log\, \eta} \end{aligned}$$

Using these we get:

$$\frac{\check{S}'^{(c)}_{0m}}{1+s'} \approx \int_{-\infty}^{\infty} dk\; t(k) v_m^{(-k)} (\sqrt{2}k) \delta(k) \left(\frac{\Re F_c(k)}{4\log\, \eta} - \frac{1}{4} \right) \frac{1}{2\eta} = 0$$

and

$$\frac{\check{S}'^{(d)}_{0m}}{1+s'} \approx \frac{1}{\sqrt{2\log \eta}}$$

Hereby the conclusion (7.35) follows.

Appendix C

Computations with dressed states

This section is devoted to the evaluation of determinants which appear in calculations involving dressed slivers. Here we deal only with the matter determinants, but the same results hold for the corresponding ghost determinants.

C.1 Evaluation of determinants

C.1.1 $\mathrm{Det}(1-\hat{\mathcal{T}}_\epsilon \mathcal{M})$

First of all we consider

$$(1-\hat{\mathcal{T}}_\epsilon \mathcal{M})^{-1}\mathcal{P} = \mathcal{K}^{-1}(1-\epsilon\mathcal{P}\mathcal{M}\mathcal{K}^{-1})^{-1}\mathcal{P} \tag{C.1}$$

This matrix can be exactly computed from

$$\mathcal{P}\mathcal{M}\mathcal{K}^{-1}\mathcal{P} = \begin{pmatrix} \kappa & \rho_1-\kappa\rho_2 \\ \rho_2-\kappa\rho_1 & \kappa \end{pmatrix}\mathcal{P} \tag{C.2}$$

We have in fact

$$(1-\epsilon\mathcal{P}\mathcal{M}\mathcal{K}^{-1})^{-1}\mathcal{P} = \sum_{n=0}^{\infty}\left(\epsilon\mathcal{P}\mathcal{M}\mathcal{K}^{-1}\right)^n\mathcal{P} \tag{C.3}$$

Using the properties of the ρ projectors, defined in the previous appendix, we can easily show that

$$\left(\epsilon\mathcal{P}\mathcal{M}\mathcal{K}^{-1}\right)^n\mathcal{P} = \frac{\epsilon^n}{(\kappa+1)^n}\begin{pmatrix} A(n) & B(n)(\rho_1-\kappa\rho_2) \\ B(n)(\rho_2-\kappa\rho_1) & A(n) \end{pmatrix}\mathcal{P} \tag{C.4}$$

where

$$A(n) = \sum_{l=0}^{\left[\frac{n}{2}\right]}(-1)^l k^{n-l}\binom{n}{2l} \tag{C.5}$$

$$B(n) = \sum_{p=0}^{\left[\frac{n-1}{2}\right]}(-1)^p k^{n-p-1}\binom{n}{2p+1} \tag{C.6}$$

Now we exchange the order of summations

$$\sum_{n=0}^{\infty}\sum_{l=0}^{\left[\frac{n}{2}\right]} = \sum_{l=0}^{\infty}\sum_{n=2l}^{\infty}$$
$$\sum_{n=0}^{\infty}\sum_{p=0}^{\left[\frac{n-1}{2}\right]} = \sum_{p=0}^{\infty}\sum_{n=2p+1}^{\infty}$$

and use the resummation formula

$$\sum_{n=l}^{\infty}\binom{n}{l}\frac{p^{n-l}}{q^n} = \frac{q}{(q-p)^{l+1}} \tag{C.7}$$

With standard algebraic manipulations, we get

$$(1-\epsilon\mathcal{P}\mathcal{M}\mathcal{K}^{-1})^{-1}\mathcal{P} = \frac{\kappa+1}{(\kappa-\epsilon\kappa+1)^2+\epsilon^2\kappa}\begin{pmatrix}\kappa-\epsilon\kappa+1 & \epsilon(\rho_1-\kappa\rho_2)\\ \epsilon(\rho_2-\kappa\rho_1) & \kappa-\epsilon\kappa+1\end{pmatrix}\mathcal{P} \tag{C.8}$$

In order to compute $\mathrm{Trln}(1-\epsilon\mathcal{P}\mathcal{M}\mathcal{K}^{-1})$ we first consider

$$\begin{aligned}
&\frac{d}{d\epsilon}\mathrm{Trln}(1-\epsilon\mathcal{P}\mathcal{M}\mathcal{K}^{-1}) = -\mathrm{Tr}\left[(1-\epsilon\mathcal{P}\mathcal{M}\mathcal{K}^{-1})^{-1}\mathcal{P}\mathcal{M}\mathcal{K}^{-1}\right]\\
&= -\mathrm{Tr}\left[\frac{\kappa+1}{(\kappa-\epsilon\kappa+1)^2+\epsilon^2\kappa}\begin{pmatrix}\kappa-\epsilon\kappa+1 & \epsilon(\rho_1-\kappa\rho_2)\\ \epsilon(\rho_2-\kappa\rho_1) & \kappa-\epsilon\kappa+1\end{pmatrix}\mathcal{P}\mathcal{M}\mathcal{K}^{-1}\right]\\
&= -\frac{\kappa+1}{(\kappa-\epsilon\kappa+1)^2+\epsilon^2\kappa}\mathrm{tr}\left[2(\kappa-\epsilon\kappa+1)P\frac{T}{1-T^2}-\epsilon P\frac{T}{1-T^2}-\epsilon\kappa P\frac{1}{1-T^2}\right]\\
&= -\frac{2(\kappa+1)}{(\kappa-\epsilon\kappa+1)^2+\epsilon^2\kappa}\left(2(\kappa-\epsilon\kappa+1)\frac{\kappa}{\kappa+1}-\epsilon\frac{\kappa}{\kappa+1}-\epsilon\kappa\frac{1}{\kappa+1}\right)\\
&= -4\frac{\kappa(\kappa+1)(1-\epsilon)}{(\kappa-\epsilon\kappa+1)^2+\epsilon^2\kappa}
\end{aligned} \tag{C.9}$$

Hence we get

$$\mathrm{Trln}(1-\epsilon\mathcal{P}\mathcal{M}\mathcal{K}^{-1}) = -4\int_0^{\epsilon} d\epsilon'\,\frac{\kappa(\kappa+1)(1-\epsilon')}{(\kappa-\epsilon'\kappa+1)^2+\epsilon'^2\kappa} = 2\ln\frac{1+(1-\epsilon)^2\kappa}{\kappa+1} \tag{C.10}$$

Collecting all the contributions we finally obtain

$$\mathrm{Det}(1-\hat{\mathcal{T}}_{\epsilon}\mathcal{M}) = \left(\frac{1+(1-\epsilon)^2\kappa}{\kappa+1}\right)^2\mathrm{Det}(1-\mathcal{T}\mathcal{M}) \tag{C.11}$$

C.1.2 $\mathrm{Det}(1-\hat{\mathcal{T}}_{\epsilon_1\epsilon_2}\mathcal{M})$

To compute this determinant we use the same strategy as before, that is we first compute

$$(1-\mathcal{P}_{\epsilon_1\epsilon_2}\mathcal{M}\mathcal{K}^{-1})^{-1}\mathcal{P}_{\epsilon_1\epsilon_2} = \begin{pmatrix}A & B(\rho_1-\kappa\rho_2)\\ D(\rho_2-\kappa\rho_1) & D\end{pmatrix}\mathcal{P}_{\epsilon_1\epsilon_2} \tag{C.12}$$

where A, B, C, D are to be determined. Moreover we have defined

$$\mathcal{P}_{\epsilon_1\epsilon_2} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix} P \tag{C.13}$$

$$\hat{\mathcal{T}}_{\epsilon_1\epsilon_2} = \begin{pmatrix} \hat{T}_{\epsilon_1} & 0 \\ 0 & \hat{T}_{\epsilon_2} \end{pmatrix} \tag{C.14}$$

The constant A, B, C, D can be easily determined by imposing $(1-\mathcal{P}_{\epsilon_1\epsilon_2}\mathcal{M}\mathcal{K}^{-1})^{-1}\mathcal{P}_{\epsilon_1\epsilon_2}$ to give back $\mathcal{P}_{\epsilon_1\epsilon_2}$ when multiplied on the left by $(1-\mathcal{P}_{\epsilon_1\epsilon_2}\mathcal{M}\mathcal{K}^{-1})$. The procedure is straightforward and gives the result

$$(1-\mathcal{P}_{\epsilon_1\epsilon_2}\mathcal{M}\mathcal{K}^{-1})^{-1}\mathcal{P}_{\epsilon_1\epsilon_2} = \frac{1}{1+(1-\epsilon_1)(1-\epsilon_2)\kappa} \begin{pmatrix} \kappa+1-\epsilon_2\kappa & \epsilon_1(\rho_1-\kappa\rho_2) \\ \epsilon_2(\rho_2-\kappa\rho_1) & \kappa+1-\epsilon_1\kappa \end{pmatrix} \mathcal{P}_{\epsilon_1\epsilon_2} \tag{C.15}$$

Now we come to the computation of the determinant

$$\mathrm{Det}(1-\hat{\mathcal{T}}_{\epsilon_1\epsilon_2}\mathcal{M}) = \mathrm{Det}(1-\mathcal{T}\mathcal{M}) \exp\left(\mathrm{Trln}(1-\mathcal{P}_{\epsilon_1\epsilon_2}\mathcal{M}\mathcal{K}^{-1})\right) \tag{C.16}$$

To compute the exponent of the second factor in the *rhs* we use the same strategy as before, namely we consider

$$\frac{d}{dx}\mathrm{Trln}(1-x\mathcal{P}_{\epsilon_1\epsilon_2}\mathcal{M}\mathcal{K}^{-1}) = -\mathrm{Tr}\left[(1-x\mathcal{P}_{\epsilon_1\epsilon_2}\mathcal{M}\mathcal{K}^{-1})^{-1}\mathcal{P}_{\epsilon_1\epsilon_2}\mathcal{M}\mathcal{K}^{-1}\right]$$

$$= -\mathrm{Tr}\left[\frac{1}{x}(1-\mathcal{P}_{x\epsilon_1,x\epsilon_2}\mathcal{M}\mathcal{K}^{-1})^{-1}\mathcal{P}_{x\epsilon_1,x\epsilon_2}\mathcal{M}\mathcal{K}^{-1}\right] \tag{C.17}$$

$$= -\mathrm{Tr}\left[\frac{1}{x}\frac{1}{1+(1-\epsilon_1)(1-\epsilon_2)\kappa} \begin{pmatrix} \kappa+1-x\epsilon_2\kappa & x\epsilon_1(\rho_1-\kappa\rho_2) \\ x\epsilon_2(\rho_2-\kappa\rho_1) & \kappa+1-x\epsilon_1\kappa \end{pmatrix} \mathcal{P}_{x\epsilon_1,x\epsilon_2}\mathcal{M}\mathcal{K}^{-1}\right]$$

$$= -2\frac{(\epsilon_1+\epsilon_2)\kappa-2x\epsilon_1\epsilon_2\kappa}{1+(1-x\epsilon_1)(1-x\epsilon_2)\kappa} \tag{C.18}$$

where the same manipulations as in (C.9) have been used. Then we perform the simple integration

$$\mathrm{Trln}(1-\mathcal{P}_{\epsilon_1\epsilon_2}\mathcal{M}\mathcal{K}^{-1}) = -2\int_0^1 dx\frac{(\epsilon_1+\epsilon_2)\kappa-2x\epsilon_1\epsilon_2\kappa}{1+(1-x\epsilon_1)(1-x\epsilon_2)\kappa} = 2\ln\left(\frac{1+(1-\epsilon_1)(1-\epsilon_2)\kappa}{\kappa+1}\right) \tag{C.19}$$

Therefore we have obtained

$$\mathrm{Det}(1-\hat{\mathcal{T}}_{\epsilon_1\epsilon_2}\mathcal{M}) = \left(\frac{1+(1-\epsilon_1)(1-\epsilon_2)\kappa}{\kappa+1}\right)^2 \mathrm{Det}(1-\mathcal{T}\mathcal{M}) \tag{C.20}$$

C.1.3 $\det(1-\hat{T}_\epsilon^2)$

We have

$$\det(1-\hat{T}_\epsilon^2) = \det(1-\hat{T}_\epsilon)\det(1+\hat{T}_\epsilon) \tag{C.21}$$

We compute the two factors separately

$$\det(1-\hat{T}_\epsilon) = \det\left(1 - \frac{\epsilon}{1-T}P\right)\det(1-T) \tag{C.22}$$

For the first factor in the *rhs* we have

$$\begin{aligned}
\det(1-\frac{\epsilon}{1-T}P) &= \exp\left(\mathrm{trln}(1-\frac{\epsilon}{1-T}P)\right)\\
&= \exp\left(-\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}(\frac{\epsilon}{1-T}P)^n\right) = \exp\left(-2\sum_{n=1}^{\infty}\frac{\epsilon^n}{n(\kappa+1)^n}\langle\xi|\frac{1}{1-T}|\xi\rangle^n\right)\\
&= \exp\left(-2\sum_{n=1}^{\infty}\frac{\epsilon^n}{n(\kappa+1)^n}(\kappa+1)^n\right) = \exp\left(2\ln(1-\epsilon)\right)\\
&= (1-\epsilon)^2
\end{aligned} \tag{C.23}$$

So we have

$$\det(1-\hat{T}_\epsilon) = (1-\epsilon)^2\det(1-T) \tag{C.24}$$

Now let's turn to the second factor in (C.21)

$$\det(1+\hat{T}_\epsilon) = \det\left(1 + \frac{\epsilon}{1+T}P\right)\det(1+T) \tag{C.25}$$

Computing as in (C.23) we obtain

$$\det\left(1+\frac{\epsilon}{1+T}P\right) = \left(\frac{\kappa+1-\epsilon(\kappa-1)}{\kappa+1}\right)^2 \tag{C.26}$$

giving the result

$$\det(1+\hat{T}_\epsilon) = \left(\frac{\kappa+1-\epsilon(\kappa-1)}{\kappa+1}\right)^2\det(1+T) \tag{C.27}$$

Collecting the two results (C.23,C.26) we get

$$\det(1-\hat{T}_\epsilon^2) = (1-\epsilon)^2\left(\frac{\kappa+1-\epsilon(\kappa-1)}{\kappa+1}\right)^2\det(1-T^2) \tag{C.28}$$

C.1.4 $\det(1-\hat{T}_{\epsilon_1}\hat{T}_{\epsilon_2})$

First of all we decompose

$$\begin{aligned}
&1-\hat{T}_{\epsilon_1}\hat{T}_{\epsilon_2} = (1-\hat{T}_{\epsilon_1})(1+\hat{T}_{\epsilon_2}) + \hat{T}_{\epsilon_1} - \hat{T}_{\epsilon_2}\\
&= (1-\hat{T}_{\epsilon_1})(1+(\epsilon_1-\epsilon_2)(1-\hat{T}_{\epsilon_1})^{-1}P(1+\hat{T}_{\epsilon_2})^{-1})(1+\hat{T}_{\epsilon_2})
\end{aligned} \tag{C.29}$$

So we have

$$\det(1-\hat{T}_{\epsilon_1}\hat{T}_{\epsilon_2}) = \det(1-\hat{T}_{\epsilon_1})\det(1+\hat{T}_{\epsilon_2})\det(1+(\epsilon_1-\epsilon_2)(1-\hat{T}_{\epsilon_1})^{-1}P(1+\hat{T}_{\epsilon_2})^{-1}) \tag{C.30}$$

We need to compute the third factor in *rhs*

$$\det\left(1+(\epsilon_1-\epsilon_2)(1-\hat{T}_{\epsilon_1})^{-1}P(1+\hat{T}_{\epsilon_2})^{-1}\right)= \tag{C.31}$$
$$=\exp\left(\mathrm{trln}(1+(\epsilon_1-\epsilon_2)(1-\hat{T}_{\epsilon_1})^{-1}P(1+\hat{T}_{\epsilon_2})^{-1})\right)$$
$$=\exp\left(-2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{\epsilon_1-\epsilon_2}{\kappa+1}\right)^n\langle\xi|(1+\hat{T}_{\epsilon_2})^{-1}(1-\hat{T}_{\epsilon_1})^{-1}|\xi\rangle^n\right)$$

where the factor 2 is to take into account the (equal) contributions of ξ and $C\xi$ which constitute P, so from now on only the contribution of ξ is needed to be considered. Then we decompose $(1+\hat{T}_{\epsilon_2})^{-1}$ and $(1-\hat{T}_{\epsilon_1})^{-1}$ as

$$(1+\hat{T}_{\epsilon_2})^{-1}=\left((1+\epsilon_2P\frac{1}{1+T})(1+T)\right)^{-1}=\frac{1}{1+T}\sum_{m=0}^{\infty}\left(\frac{-\epsilon_2}{\kappa+1}\right)^m\left(|\xi\rangle\langle\xi|\frac{1}{1+T}\right)^m$$
$$(1-\hat{T}_{\epsilon_1})^{-1}=\left((1-T)(1+\frac{1}{1-T}\epsilon_1P)\right)^{-1}=\sum_{p=0}^{\infty}\left(\frac{\epsilon_1}{\kappa+1}\right)^p\left(\frac{1}{1-T}(|\xi\rangle\langle\xi|\right)^p\frac{1}{1-T}$$

So we get

$$\langle\xi|(1+\hat{T}_{\epsilon_2})^{-1}(1-\hat{T}_{\epsilon_1})^{-1}|\xi\rangle^n$$
$$=\left(\sum_{m=0}^{\infty}\left(\frac{-\epsilon_2}{\kappa+1}\right)^m\langle\xi|\frac{1}{1+T}|\xi\rangle^m\langle\xi|\frac{1}{1-T^2}|\xi\rangle\sum_{p=0}^{\infty}\left(\frac{\epsilon_1}{\kappa+1}\right)^p\langle\xi|\frac{1}{1-T}|\xi\rangle^p\right)^n$$
$$=\left(\frac{\kappa+1}{(\kappa+1-\epsilon_2(\kappa-1))(1-\epsilon_1)}\right)^n \tag{C.32}$$

plugging this in (C.31), we get

$$\det\left(1+(\epsilon_1-\epsilon_2)(1-\hat{T}_{\epsilon_1})^{-1}P(1+\hat{T}_{\epsilon_2})^{-1}\right) \tag{C.33}$$
$$=\exp\left(-2\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\epsilon_2-\epsilon_1}{(1-\epsilon_1)(\kappa+1-\epsilon_2(\kappa-1)}\right)^n\right)$$
$$=\left(1-\frac{\epsilon_2-\epsilon_1}{(1-\epsilon_1)(\kappa+1-\epsilon_2(\kappa-1)}\right)^2 \tag{C.34}$$

From (C.30), using (C.24, C.27), we finally get

$$\det(1-\hat{T}_{\epsilon_1}\hat{T}_{\epsilon_2})=\det(1-T^2)\left(1-(\epsilon_1+\epsilon_2)\frac{\kappa}{\kappa+1}+\epsilon_1\epsilon_2\frac{\kappa-1}{\kappa+1}\right)^2 \tag{C.35}$$
$$=\det(1-T^2)\left[\frac{\epsilon_1\epsilon_2}{\kappa+1}\left(1-\frac{1}{\epsilon_1\star\epsilon_2}\right)\right]^2$$
$$=\det(1-\hat{T}^2_{\epsilon_1})^{\frac{1}{2}}\det(1-\hat{T}^2_{\epsilon_2})^{\frac{1}{2}}\left(1-\frac{\epsilon_2-\epsilon_1}{(1-\epsilon_1)(\kappa+1-\epsilon_2(\kappa-1)}\right)\left(1-\frac{\epsilon_1-\epsilon_2}{(1-\epsilon_2)(\kappa+1-\epsilon_1(\kappa-1)}\right)$$

Note that the last two factors in *rhs* of the last line approach 1 as $\epsilon_1\to\epsilon_2$.

C.2 Limit prescriptions

C.2.1 Double limit

In this appendix we analyse various limits of the quantity

$$\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}\rangle = \frac{\hat{\mathcal{N}}_{\epsilon_1}\hat{\mathcal{N}}_{\epsilon_2}}{\det(1-\hat{S}_{\epsilon_1}\hat{S}_{\epsilon_2})}\langle 0|0\rangle \tag{C.36}$$

when $\epsilon_1, \epsilon_2 \to 1$. We recall that $0 \le \epsilon_1, \epsilon_2 \le 1$.

Since $\det(1-\hat{S}_{\epsilon_1}\hat{S}_{\epsilon_2}) = \det(1-\hat{S}_{\epsilon_2}\hat{S}_{\epsilon_1})$ and

$$\det(1-\hat{S}_{\epsilon_1}\hat{S}_{\epsilon_2}) = \left(\det(1-\hat{T}_{\epsilon_1})\det(1+\hat{T}_{\epsilon_2})\det\left(1+(\epsilon_1-\epsilon_2)P\frac{1}{(1-\hat{T}_{\epsilon_1})(1+\hat{T}_{\epsilon_2})}\right)\right)^{\frac{1}{2}} \tag{C.37}$$

it is convenient to symmetrize the result. One gets

$$\det(1-\hat{S}_{\epsilon_1}\hat{S}_{\epsilon_2}) = \left(\det(1-\hat{T}^2_{\epsilon_1})\det(1-\hat{T}^2_{\epsilon_2})\right)^{\frac{1}{2}} \tag{C.38}$$

$$\cdot\left(\det\left(1+(\epsilon_1-\epsilon_2)P\frac{1}{(1-\hat{T}_{\epsilon_1})(1+\hat{T}_{\epsilon_2})}\right)\det\left(1+(\epsilon_2-\epsilon_1)P\frac{1}{(1-\hat{T}_{\epsilon_2})(1+\hat{T}_{\epsilon_1})}\right)\right)^{\frac{1}{2}}$$

Using the results of Appendix B.4 this can be rewritten as

$$\det(1-\hat{S}_{\epsilon_1}\hat{S}_{\epsilon_2}) = \det(1-\hat{T}^2_{\epsilon_1})^{\frac{1}{2}}\det(1-\hat{T}^2_{\epsilon_2})^{\frac{1}{2}}$$
$$\cdot\left(1-\frac{\epsilon_2-\epsilon_1}{(1-\epsilon_1)(\kappa+1-\epsilon_2(\kappa-1)}\right)\left(1-\frac{\epsilon_1-\epsilon_2}{(1-\epsilon_2)(\kappa+1-\epsilon_1(\kappa-1)}\right)$$

Therefore, collecting the previous results,

$$\frac{1}{\langle 0|0\rangle}\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}\rangle = \left(\frac{\hat{\mathcal{N}}_{\epsilon_1}}{\sqrt{\det(1-\hat{S}^2_{\epsilon_1})}}\right)^{\frac{D}{2}}\left(\frac{\hat{\mathcal{N}}_{\epsilon_2}}{\sqrt{\det(1-\hat{S}^2_{\epsilon_2})}}\right)^{\frac{D}{2}}$$
$$\cdot\left(\frac{(1-\epsilon_1)(1-\epsilon_2)(\kappa+1-\epsilon_1(\kappa-1))(\kappa+1-\epsilon_2(k-1))}{(\kappa+1-(\epsilon_1+\epsilon_2)\kappa+\epsilon_1\epsilon_2(\kappa-1))^2}\right)^{\frac{D}{2}} \tag{C.39}$$

When ϵ_1 and ϵ_2 are in the vicinity of 1, this simplifies as follows

$$\left(\frac{\det(1-\Sigma\mathcal{V})}{\sqrt{\det(1-S^2)}}\right)^{D}\left(\frac{1}{4(\kappa+1)^2}\right)^{\frac{D}{2}}\left(\frac{4}{(\kappa(1-\epsilon_1)(1-\epsilon_2)+1-\epsilon_1\epsilon_2)^2}\right)^{\frac{D}{2}}+\ldots \tag{C.40}$$

where dots denote non–leading terms. It is useful to change parametrization of ϵ_1, ϵ_2 as follows

$$1-\epsilon_1 = r\cos\theta \qquad 1-\epsilon_2 = r\sin\theta, \qquad 0\le\theta\le\pi/2 \tag{C.41}$$

Then (C.40) becomes

$$\left(\frac{\det(1-\Sigma\mathcal{V})}{\sqrt{\det(1-S^2)}}\right)^{D}\left(\frac{1}{(\kappa+1)^2}\right)^{\frac{D}{2}}\left(\frac{1}{r^2}\frac{1}{(\sin\theta+\cos\theta)^2}\right)^{\frac{D}{2}}+\ldots \tag{C.42}$$

The function $(\sin\theta+\cos\theta)^{-2}$ varies between 1 and 1/2, with a minimum at $\theta=\pi/4$, which corresponds to $\epsilon_1=\epsilon_2=\epsilon$ ($r=\sqrt{2}(1-\epsilon)$), and maxima at $\theta=0,\pi/2$, which correspond to $\epsilon_1=r,\epsilon_2=0$ and $\epsilon_1=0,\epsilon_2=r$.

These are the two possibilities considered in section 5. The first corresponds to $\epsilon_1=\epsilon_2=\epsilon$, the second corresponds to the ordered limit. In between there are of course infinite many possibilities, giving rise to different rescalings of the number s.

C.2.2 Triple limit

We discuss here the *rhs* of eq.(4.70). We start with calculating an explicit formula for $\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}*\hat{\Xi}_{\epsilon_3}\rangle$,

$$\begin{aligned}
&\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}*\hat{\Xi}_{\epsilon_3}\rangle=\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2\star\epsilon_3}\rangle=\left(\frac{\hat{\mathcal{N}}_{\epsilon_1}\hat{\mathcal{N}}_{\epsilon_2\star\epsilon_3}}{\sqrt{1-\hat{T}_{\epsilon_1}\hat{T}_{\epsilon_2\star\epsilon_3}}}\right)^D\\
&\sim\left(\frac{\mathrm{Det}(1-\mathcal{TM})}{\sqrt{1-T^2}}\right)^D\frac{1}{(1+\kappa)^D}\left(1+\kappa-(\epsilon_1+\epsilon_2\star\epsilon_3)\kappa+\epsilon_1\epsilon_2\star\epsilon_3(\kappa-1)\right)^{-D}\\
&\sim\left(\frac{\mathrm{Det}(1-\mathcal{TM})}{\sqrt{1-T^2}}\right)^D\frac{1}{(1+\kappa)^D}\\
&\cdot\left(\kappa^2(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)+\kappa(2-\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_1\epsilon_2\epsilon_3)+1-\epsilon_1\epsilon_2\epsilon_3\right)^{-D}
\end{aligned}\tag{C.43}$$

where we have kept only the dominant term for $\epsilon_1,\epsilon_2,\epsilon_3$ near 1. Now let us introduce the parametrization

$$1-\epsilon_1=r\cos\theta,\qquad 1-\epsilon_2=r\sin\theta\cos\varphi,\qquad 1-\epsilon_2=r\sin\theta\sin\varphi\tag{C.44}$$

where $0\leq\theta\leq\pi/2$, $0\leq\varphi\leq\pi/2$. Then (C.43) becomes (keeping only the dominant term)

$$\begin{aligned}
&\langle\hat{\Xi}_{\epsilon_1}|\hat{\Xi}_{\epsilon_2}*\hat{\Xi}_{\epsilon_3}\rangle\\
&\sim\left(\frac{\mathrm{Det}(1-\mathcal{TM})}{\sqrt{1-T^2}}\right)^D\frac{1}{(1+\kappa)^D}\left(\frac{1}{r^2(\cos\theta+\sin\theta(\cos\varphi+\sin\varphi))^2}\right)^{\frac{D}{2}}
\end{aligned}\tag{C.45}$$

The function $\frac{1}{(\cos\theta+\sin\theta(\cos\varphi+\sin\varphi))^2}$ varies between a minimum of 1/3, when $\epsilon_1=\epsilon_2=\epsilon_3=\epsilon$ ($r=\sqrt{3}(1-\epsilon)$), and a maximum of 1. Thus it is clear that for $\epsilon_1=\epsilon_2=\epsilon_3$ eq.(4.73) cannot be satisfied. On the other hand there are many ways to satisfy

$$\frac{1}{\cos\theta+\sin\theta(\cos\varphi+\sin\varphi)}=\frac{1}{\cos\phi+\sin\phi}\tag{C.46}$$

in which case (C.45) reduces to (C.42). The simplest way is to set $\varphi=0,\theta=\phi$ or $\theta=\pi/2,\varphi=\phi$. These correspond to ordered limits.

C.3 Derivation of ghost product for ghost dressed sliver

Here we sketch a derivation of the $*_g$ product of two states of the form (4.86) given in eq. (4.90). We need to calculate

$$|\widetilde{\Xi}_{\tilde{\epsilon}}\rangle *_g |\widetilde{\Xi}_{\tilde{\eta}}\rangle = {}_1\langle\widetilde{\Xi}_{\tilde{\epsilon}}|\,{}_2\langle\widetilde{\Xi}_{\tilde{\eta}}|\widetilde{V}_3\rangle \;, \tag{C.47}$$

where the ghost part of the 3-strings vertex $|\widetilde{V}_3\rangle$ is given in (4.79). Using the rules for bpz-conjugation we obtain

$$\langle\widetilde{\Xi}_{\tilde{\epsilon}}| = \widetilde{N}_{\tilde{\epsilon}}\,\langle 0|c_1^\dagger e^{-c\widetilde{S}_{\tilde{\epsilon}}b} \;. \tag{C.48}$$

Plugging this and (4.79) in (C.47), and following the steps outlined in [23], one obtains

$$\begin{aligned}|\widetilde{\Xi}_{\tilde{\epsilon}}\rangle *_g |\widetilde{\Xi}_{\tilde{\eta}}\rangle &= \widetilde{\mathcal{N}}_{\tilde{\epsilon}}\widetilde{\mathcal{N}}_{\tilde{\eta}}\,\det(1-\widetilde{\mathcal{T}}_{\tilde{\epsilon}\tilde{\eta}}\widetilde{\mathcal{M}}) \\ &\times\left\{1+c^\dagger\left[\widetilde{\mathbf{v}}_0+\left(\widetilde{V}_+,\widetilde{V}_-\right)(1-\widetilde{\mathcal{T}}_{\tilde{\epsilon}\tilde{\eta}}\widetilde{\mathcal{M}})^{-1}\widetilde{\mathcal{T}}_{\tilde{\epsilon}\tilde{\eta}}\begin{pmatrix}\widetilde{\mathbf{v}}_+\\ \widetilde{\mathbf{v}}_-\end{pmatrix}\right]b_0\right\}e^{c^\dagger C\widetilde{T}_{\tilde{\epsilon}}*\widetilde{T}_{\tilde{\eta}}b^\dagger}c_0c_1|0\rangle \;.\end{aligned} \tag{C.49}$$

Here the summations over mode indexes with positive values are understood. $\widetilde{\mathcal{T}}_{\tilde{\epsilon}\tilde{\eta}}$ and $\widetilde{\mathcal{M}}$ are defined as in (4.43) and (4.24), but with tildes. We also define

$$\widetilde{T}_{\tilde{\epsilon}} * \widetilde{T}_{\tilde{\eta}} = \widetilde{X} + \left(\widetilde{X}_+,\widetilde{X}_-\right)(1-\widetilde{\mathcal{T}}_{\tilde{\epsilon}\tilde{\eta}}\widetilde{\mathcal{M}})^{-1}\widetilde{\mathcal{T}}_{\tilde{\epsilon}\tilde{\eta}}\begin{pmatrix}\widetilde{X}_-\\ \widetilde{X}_+\end{pmatrix} \;. \tag{C.50}$$

Now one should observe that this formula is the same as (4.42), and as tilded matrices satisfy the same algebraic relations as the untilded ones , the result has the same form, i.e., we obtain

$$\widetilde{T}_{\tilde{\epsilon}} * \widetilde{T}_{\tilde{\eta}} = \widetilde{T}_{\tilde{\epsilon}\star\tilde{\eta}} \;. \tag{C.51}$$

This constitutes, after using (C.20), the proof of eq. (4.88) for the reduced star product.

Now we must consider the part including the b_0 mode. After lengthier but straightforward manipulations, using the formulas from appendix A and sections 2, 3 and 4, we obtain that the expression inside square brackets in (C.49) can be written as

$$\widetilde{\mathbf{v}}_0+\left(\widetilde{V}_+,\widetilde{V}_-\right)(1-\widetilde{\mathcal{T}}_{\tilde{\epsilon}\tilde{\eta}}\widetilde{\mathcal{M}})^{-1}\widetilde{\mathcal{T}}_{\tilde{\epsilon}\tilde{\eta}}\begin{pmatrix}\widetilde{\mathbf{v}}_+\\ \widetilde{\mathbf{v}}_-\end{pmatrix} = C(1-\widetilde{T}_{\tilde{\epsilon}\star\tilde{\eta}})\mathbf{f} \;, \tag{C.52}$$

where $\mathbf{f}=\{f_n\}$ is the vector defined in appendix A. Using this in (C.49) we get

$$|\widetilde{\Xi}_{\tilde{\epsilon}}\rangle *_g |\widetilde{\Xi}_{\tilde{\eta}}\rangle = \frac{\widetilde{\mathcal{N}}_{\tilde{\epsilon}}\widetilde{\mathcal{N}}_{\tilde{\eta}}}{\widetilde{\mathcal{N}}_{\tilde{\epsilon}\star\tilde{\eta}}}\left[\frac{1+(1-\tilde{\epsilon})(1-\tilde{\eta})\tilde{\kappa}}{\tilde{\kappa}+1}\right]^2\det(1-\widetilde{T}\widetilde{\mathcal{M}})\left[c_0+c^\dagger C(1-\widetilde{T}_{\tilde{\epsilon}\star\tilde{\eta}})\mathbf{f}\right]|\widetilde{\Xi}_{\tilde{\epsilon}\star\tilde{\eta}}\rangle \;. \tag{C.53}$$

On the other hand, one easily shows that

$$\mathcal{Q}\,|\widetilde{\Xi}_{\tilde{\epsilon}}\rangle = \left(c_0+\sum_{n=1}^{\infty}f_n(c_n+(-1)^n c_n^\dagger)\right)|\widetilde{\Xi}_{\tilde{\epsilon}}\rangle = \left[c_0+c^\dagger C(1-\widetilde{T}_{\tilde{\epsilon}})\mathbf{f}\right]|\widetilde{\Xi}_{\tilde{\epsilon}}\rangle \;. \tag{C.54}$$

Using this in (C.53) finally one gets (4.90).

Appendix D

Excitations of the dressed sliver

This appendix is devoted to summarize the copious computations needed to determine the open string spectrum on the dressed sliver solution

D.1 Solving for $\mathbf{t}_+$ and $\mathbf{t}_-$

To solve for $\mathbf{t} = \mathbf{t}_+ + \mathbf{t}_-$ in the LEOM in full generality, we reintroduce the parameter ϵ in the equation of motion (5.6). This means deforming it as follows

$$\exp[-\mathbf{t}' a^\dagger \hat{p}]|\hat{\Xi}_{e*\epsilon}\rangle = |\hat{\Xi}_\epsilon\rangle * (\exp[-\mathbf{t} a^\dagger \hat{p}]|\hat{\Xi}_e\rangle) + (\exp[-\mathbf{t} a^\dagger \hat{p}]|\hat{\Xi}_e\rangle) * |\hat{\Xi}_\epsilon\rangle \tag{D.1}$$

This seems to be a sensible deformation of (5.6), since we know that, as $\epsilon \to 1$, $\hat{\Xi}_{e*\epsilon} \to \hat{\Xi}_e$. As for $\mathbf{t}'$, this deformation makes sense only if $\mathbf{t}' \to \mathbf{t}$ as $\epsilon \to 1$. This is indeed what happens.

In the following we will find a solution to (D.1) and then take the limit for $\epsilon \to 1$.

$$\mathbf{t}'_+ = \mathbf{v}_0 - \mathbf{v}_- + (X_+, X_-)\, \hat{\mathcal{K}}^{-1}_{\epsilon e}\, \hat{\mathcal{T}}_{\epsilon e} \begin{pmatrix} \mathbf{v}_- - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_0 \end{pmatrix} + (X_+, X_-)\, \hat{\mathcal{K}}^{-1}_{\epsilon e} \begin{pmatrix} 0 \\ \mathbf{t}_+ \end{pmatrix} \tag{D.2}$$

$$\mathbf{t}'_- = (X_+, X_-)\, \hat{\mathcal{K}}^{-1}_{\epsilon e} \begin{pmatrix} 0 \\ \mathbf{t}_- \end{pmatrix} \tag{D.3}$$

We rewrite eq.(D.2) in a more explicit form, using the methods and results of Appendix B of [3]. In particular we need the formula

$$(1 - \mathcal{P}_{\epsilon e}\mathcal{M}\mathcal{K}^{-1})^{-1}\mathcal{P}_{\epsilon e} = \frac{1}{B_{e\epsilon}} \begin{pmatrix} e f_e & \epsilon(\rho_1 - \kappa\rho_2) \\ e(\rho_2 - \kappa\rho_1) & \epsilon f_\epsilon \end{pmatrix} \mathcal{P}_{\epsilon e} \tag{D.4}$$

where

$$\mathcal{P}_{\epsilon e} = \begin{pmatrix} \epsilon & 0 \\ 0 & e \end{pmatrix} P\,, \qquad B_{e\epsilon} = 1 + (1-e)(1-\epsilon)\kappa\,.$$

Then eq.(5.27) can be rewritten as follows

$$\mathbf{t}'_+ = \mathbf{v}_0 - \mathbf{v}_- + (X_+, X_-)\, \mathcal{K}^{-1}\, \mathcal{T} \begin{pmatrix} \mathbf{v}_- - \mathbf{v}_+ \\ \mathbf{v}_+ - \mathbf{v}_0 \end{pmatrix} + (X_+, X_-)\, \mathcal{K}^{-1} \begin{pmatrix} 0 \\ \mathbf{t}_+ \end{pmatrix}$$

$$+\frac{1}{B_{\epsilon e}}(\rho_1,\rho_2)\begin{pmatrix} ef_e & \epsilon(\rho_1-\kappa\rho_2)\\ e(\rho_2-\kappa\rho_1) & \epsilon f_\epsilon\end{pmatrix}\mathcal{P}_{\epsilon e}\cdot \tag{D.5}$$
$$\cdot\left[\begin{pmatrix}\frac{1}{1-T^2} & \frac{TX_+}{(1+T)(1-X)}\\ \frac{TX_-}{(1+T)(1-X)} & \frac{1}{1-T^2}\end{pmatrix}\begin{pmatrix}3\frac{1-T}{1+T}(\rho_2-\rho_1)|\mathbf{v}_0\rangle\\ -\frac{3}{1+T}(\rho_2+T\rho_1)|\mathbf{v}_0\rangle\end{pmatrix}\right.$$
$$\left.+\begin{pmatrix}\frac{T}{1-T^2} & \frac{X_+}{(1+T)(1-X)}\\ \frac{X_-}{(1+T)(1-X)} & \frac{T}{1-T^2}\end{pmatrix}\begin{pmatrix}0\\ \mathbf{t}_+\end{pmatrix}\right]$$

Carrying out the algebra one finds

$$\mathbf{t}'_+=\rho_2\mathbf{t}_++\rho_1\mathbf{t}_0+\frac{1}{\kappa+f_\epsilon f_e}\quad\Big[(1-f_\epsilon)|\xi\rangle\,\langle\xi|\frac{1}{1-T^2}|\mathbf{t}_0\rangle-|C\xi\rangle\langle\xi|\frac{f_e+T}{1-T^2}|\mathbf{t}_0\rangle$$
$$+(f_\epsilon-1)|\xi\rangle\,\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle+|C\xi\rangle\langle\xi|\frac{f_e+T}{1-T^2}|\mathbf{t}_+\rangle\Big],$$

Applying now C to both sides of this equation and summing the two we get a C–symmetric equation.

$$2\mathbf{t}'_+=\mathbf{t}_++\mathbf{t}_0+\quad\frac{1}{\kappa+f_\epsilon f_e}\Big[(1-f_\epsilon)|\xi+C\xi\rangle\,\langle\xi|\frac{1}{1-T^2}|\mathbf{t}_0\rangle-|\xi+C\xi\rangle\langle C\xi|\frac{f_e+T}{1-T^2}|\mathbf{t}_0\rangle$$
$$+(f_\epsilon-1)|\xi+C\xi\rangle\,\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle+|\xi+C\xi\rangle\langle\xi|\frac{f_e+T}{1-T^2}|\mathbf{t}_+\rangle\Big] \tag{D.6}$$

Taking the difference we get instead

$$0=(\rho_2-\rho_1)(\mathbf{t}_+-\mathbf{t}_0)+\quad\frac{1}{\kappa+f_\epsilon f_e}\Big[(1-f_\epsilon)|\xi-C\xi\rangle\,\langle\xi|\frac{1}{1-T^2}|\mathbf{t}_0\rangle+|\xi-C\xi\rangle\langle C\xi|\frac{f_e+T}{1-T^2}|\mathbf{t}_0\rangle$$
$$+(f_\epsilon-1)|\xi-C\xi\rangle\,\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle-|\xi-C\xi\rangle\langle\xi|\frac{f_e+T}{1-T^2}|\mathbf{t}_+\rangle\Big] \tag{D.7}$$

Recalling that $(\rho_1-\rho_2)^2=1$, we multiply the last equation by $\rho_1-\rho_2$ and obtain

$$\mathbf{t}_+=\mathbf{t}_0-\frac{1}{\kappa+f_\epsilon f_e}\quad\Big[(1-f_\epsilon)\,\langle\xi|\frac{1}{1-T^2}|\mathbf{t}_0\rangle+\langle C\xi|\frac{f_e+T}{1-T^2}|\mathbf{t}_0\rangle$$
$$+(f_\epsilon-1)\,\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle-\langle\xi|\frac{f_e+T}{1-T^2}|\mathbf{t}_+\rangle\Big]\,|\xi+C\xi\rangle \tag{D.8}$$

The solution to this equation is clearly of the form $\mathbf{t}=\mathbf{t}_0+H|\xi+C\xi\rangle$, for some constant H. The latter can be determined by plugging this ansatz in (D.8). One easily gets

$$\mathbf{t}_+\;=\;\mathbf{t}_0+\frac{1}{\kappa+f_e}|\xi+C\xi\rangle\langle\xi|\frac{1}{1+T}|\mathbf{t}_0\rangle \tag{D.9}$$

Now we can replace this solution back into (D.6). One easily obtains

$$\mathbf{t}'_+\;=\;\mathbf{t}_0+\frac{1}{\kappa+f_\epsilon f_e}|\xi+C\xi\rangle\langle\xi|\frac{1}{1+T}|\mathbf{t}_0\rangle \tag{D.10}$$

We see that as $\epsilon\to 1$, $\mathbf{t}'_+\to\mathbf{t}_+$.

As for (D.3) we proceed in the same way. From the difference equation we obtain

$$M_-|\mathbf{t}_-\rangle\;\equiv\;\left[1+\frac{1}{\kappa+f_\epsilon f_e}\,|\xi-C\xi\rangle\left((f_\epsilon-1)\langle\xi|\frac{T}{1-T^2}-\langle\xi|\frac{f_e+T}{1-T^2}\right)\right]|\mathbf{t}_-\rangle\;=\;0 \tag{D.11}$$

The solution must be in the kernel of the operator M_- and must have the form

$$|\mathbf{t}_-\rangle = \beta\,|(1-C)\xi\rangle \tag{D.12}$$

for some constant β. Plugging this in the previous equation we find

$$M_-|\mathbf{t}_-\rangle \;=\; \beta\frac{(f_\epsilon-1)(f_e+\kappa)}{\kappa+f_\epsilon f_e}|\xi - C\xi\rangle$$

Therefore, (D.12) solves (D.11) either when $f_\epsilon = 1$ ($\epsilon = 1$), or when $f_e = -\kappa$ ($e \to \infty$) and $f_\epsilon \neq 1$. We are interested here in the first case. Putting $f_\epsilon = 1$ and (D.12) in (D.3) we obtain $\mathbf{t}'_- = \mathbf{t}_-$ for any β and f_e.

D.2 Calculating G

Let us first compute G with $\mathbf{t} = \mathbf{t}_+$ starting from eq.(5.34). Our procedure consists in separating the ξ–independent part from the rest. The latter corresponds to Hata et al.'s calculation, [23, 61, 62]. For instance

$$\begin{aligned}(\mathbf{v}_+-\mathbf{v}_-,\mathbf{v}_--\mathbf{v}_0)\,\hat{\mathcal{K}}_{\epsilon e}^{-1}\,\hat{\mathcal{T}}_{\epsilon e}\begin{pmatrix}\mathbf{v}_--\mathbf{v}_+\\ \mathbf{v}_+-\mathbf{v}_0\end{pmatrix} &= (\mathbf{v}_+-\mathbf{v}_-,\mathbf{v}_--\mathbf{v}_0)\,\mathcal{K}^{-1}\,\mathcal{T}\begin{pmatrix}\mathbf{v}_--\mathbf{v}_+\\ \mathbf{v}_+-\mathbf{v}_0\end{pmatrix} \qquad \text{(D.13)}\\ +(\mathbf{v}_+-\mathbf{v}_-,\mathbf{v}_--\mathbf{v}_0)\,\mathcal{K}^{-1}\,\frac{1}{B_{\epsilon e}}&\begin{pmatrix} ef_e & \epsilon(\rho_1-\kappa\rho_2)\\ e(\rho_2-\kappa\rho_1) & \epsilon f_\epsilon\end{pmatrix}\mathcal{P}_{\epsilon e}(1+\mathcal{M}\mathcal{K}^{-1}\mathcal{T})\begin{pmatrix}\mathbf{v}_--\mathbf{v}_+\\ \mathbf{v}_+-\mathbf{v}_0\end{pmatrix}\end{aligned}$$

where again $B_{\epsilon e} = 1 + (1-e)(1-\epsilon)\kappa$. The first piece in the RHS is the ξ–independent part. Carrying out the algebra one gets the following result (D.13)

$$\begin{aligned}(\mathbf{v}_+-\mathbf{v}_-,\mathbf{v}_--\mathbf{v}_0)\,\hat{\mathcal{K}}_{\epsilon e}^{-1}\,\hat{\mathcal{T}}_{\epsilon e}\begin{pmatrix}\mathbf{v}_--\mathbf{v}_+\\ \mathbf{v}_+-\mathbf{v}_0\end{pmatrix} &= 3\,\langle\mathbf{t}_0|\frac{T(2T-1)}{(T+1)^2(T-1)}|\mathbf{v}_0\rangle \qquad \text{(D.14)}\\ +\frac{2}{B_{\epsilon e}}\Big[\langle\mathbf{t}_0|\frac{1}{1-T^2}|\xi\rangle&\Big(e(1-\epsilon)\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_0\rangle - \epsilon\langle\xi|\frac{1}{1-T^2}|\mathbf{t}_0\rangle\Big)\Big]\end{aligned}$$

Proceeding in the same way with the third term in (5.34) we find

$$\begin{aligned}(\mathbf{v}_+-\mathbf{v}_-,\mathbf{v}_--\mathbf{v}_0)\,\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix}0\\ \mathbf{t}_+\end{pmatrix} &= \frac{1}{2}\langle\mathbf{t}_0|\frac{1}{1-T}|\mathbf{t}_0\rangle + \frac{1}{B_{\epsilon e}}\Big[e(\epsilon-1)\langle\mathbf{t}_0|\frac{T}{1-T^2}|\xi\rangle\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle\\ +\langle\mathbf{t}_0|\frac{1}{1-T^2}|\xi\rangle&\Big((\epsilon-e(1-\epsilon))\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle + \epsilon\langle\xi|\frac{1}{1-T^2}|\mathbf{t}_+\rangle\Big)\Big] \qquad \text{(D.15)}\end{aligned}$$

Similarly for the last term on the RHS of (5.34) we find

$$\begin{aligned}(0,\mathbf{t}_+)\,\mathcal{M}\,\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix}0\\ \mathbf{t}_+\end{pmatrix} &= \langle\mathbf{t}_0|\frac{T}{1-T^2}|\mathbf{t}_0\rangle + \qquad \text{(D.16)}\\ +\frac{2}{B_{\epsilon e}}\Big[e(1-\epsilon)\langle\mathbf{t}_+|\frac{T}{1-T^2}|\xi\rangle\langle\xi|&\frac{T}{1-T^2}|\mathbf{t}_+\rangle - \epsilon\langle\mathbf{t}_+|\frac{1}{1-T^2}|\xi\rangle\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle\Big]\end{aligned}$$

Now we turn to the terms containing the twist–odd part. We need

$$\begin{aligned}-2(\mathbf{v}_+-\mathbf{v}_-,\mathbf{v}_--\mathbf{v}_0)\,\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix}0\\ \mathbf{t}_-\end{pmatrix} - (0,\mathbf{t}_+)\,\mathcal{M}\,\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix}0\\ \mathbf{t}_-\end{pmatrix} + (0,\mathbf{t}_-)\,\mathcal{M}\,\hat{\mathcal{K}}_{\epsilon e}^{-1}\begin{pmatrix}0\\ \mathbf{t}_+\end{pmatrix}\\ = -\frac{\beta(1-\epsilon)}{1+(1-\epsilon)(1-e)\kappa}\Big[(2+2\kappa-\epsilon\kappa)\langle\mathbf{t}_0|\frac{1}{1+T}|\xi\rangle + e\kappa\langle\mathbf{t}_0|\frac{1}{1-T}|\xi\rangle\Big] \qquad \text{(D.17)}\end{aligned}$$

and also

$$(0, \mathbf{t}_-)\,\mathcal{M}\,\hat{\mathcal{K}}_{\epsilon e}^{-1} \begin{pmatrix} 0 \\ \mathbf{t}_- \end{pmatrix} = 2\beta^2\kappa \frac{(1-\epsilon)(\kappa+1)}{1+(1-\epsilon)(1-e)\kappa} \tag{D.18}$$

Using above formulae in (5.34) and (5.35) one obtains (5.36) and (5.37), respectively.

D.3 Formulas for star products in LEOM

In this Appendix we explicitly write down some formulas which are needed in order to evaluate the star products in the LEOM when the involved state is of the type (5.4) with a nontrivial polynomial $\mathcal{P}$, or, in other words, is the product of a tachyon–like state times a polynomial of the creation operators like (5.39). The best course in this case is to introduce the state (5.11), which depends on the variable vector β^μ, compute the star products of this state with the dressed sliver and then differentiate with respect to β^μ, setting $\beta^\mu = 0$ afterwards, in such a way as to 'pull down' the desired monomials of the type (5.39). The calculation is straightforward and the relevant results for the matter part are recorded in the following formulas (where, for simplicity, we have set $\epsilon = 1$)

$$\begin{aligned} &(\ldots\langle\zeta a_\mu^\dagger\rangle\ldots)|\hat{\varphi}_e(\mathbf{t},p)\rangle * |\hat{\Xi}\rangle = \\ &\quad = (\ldots\langle -\zeta\frac{\partial}{\partial\beta^\mu}\rangle\ldots)\exp\left[-\frac{1}{2}\mathcal{A}_1 - \mathcal{B}_1 - p\cdot(\mathcal{C}_1+\mathcal{D}_1)\right]|\hat{\varphi}_e(\mathbf{t},p)\rangle\Big|_{\beta=0} \end{aligned} \tag{D.19}$$

$$\begin{aligned} &|\hat{\Xi}\rangle * (\ldots\langle\zeta a_\mu^\dagger\rangle\ldots)|\hat{\varphi}_e(\mathbf{t},p)\rangle = \\ &\quad = (\ldots\langle -\zeta\frac{\partial}{\partial\beta^\mu}\rangle\ldots)\exp\left[-\frac{1}{2}\mathcal{A}_2 - \mathcal{B}_2 - p\cdot(\mathcal{C}_2+\mathcal{D}_2)\right]|\hat{\varphi}_e(\mathbf{t},p)\rangle\Big|_{\beta=0} \end{aligned} \tag{D.20}$$

where

$$\mathcal{A}_1 \equiv (\beta,0)\,\mathcal{M}\,\hat{\mathcal{K}}_{e1}^{-1} \begin{pmatrix} C\beta \\ 0 \end{pmatrix} \tag{D.21}$$

$$= \langle\beta|\frac{T}{1-T^2}|C\beta\rangle - \langle\beta|\frac{1}{1-T^2}|C\xi\rangle\langle\xi|\frac{T}{1-T^2}|\beta\rangle - \langle\beta|\frac{T}{1-T^2}|\xi\rangle\langle\xi|\frac{1}{1-T^2}|C\beta\rangle$$

$$\mathcal{A}_2 \equiv (0,\beta)\,\mathcal{M}\,\hat{\mathcal{K}}_{1e}^{-1} \begin{pmatrix} 0 \\ C\beta \end{pmatrix} \tag{D.22}$$

$$= \langle\beta|\frac{T}{1-T^2}|C\beta\rangle - \langle\beta|\frac{1}{1-T^2}|\xi\rangle\langle\xi|\frac{T}{1-T^2}|C\beta\rangle - \langle C\beta|\frac{T}{1-T^2}|\xi\rangle\langle\xi|\frac{1}{1-T^2}|\beta\rangle$$

$$\mathcal{B}_1 \equiv a^\dagger(V_+,V_-)\,\hat{\mathcal{K}}_{e1}^{-1} \begin{pmatrix} C\beta \\ 0 \end{pmatrix} = \langle a^\dagger\rho_2\beta\rangle + \frac{1}{\kappa+f_e}\langle a^\dagger C\xi\rangle\langle\xi|\frac{T+f_e}{1-T^2}|C\beta\rangle \tag{D.23}$$

$$\mathcal{B}_2 \equiv a^\dagger(V_+,V_-)\,\hat{\mathcal{K}}_{1e}^{-1} \begin{pmatrix} 0 \\ C\beta \end{pmatrix} = \langle a^\dagger\rho_1\beta\rangle + \frac{1}{\kappa+f_e}\langle a^\dagger\xi\rangle\langle\xi|\frac{T+f_e}{1-T^2}|\beta\rangle \tag{D.24}$$

$$\mathcal{C}_1 \equiv (\mathbf{t},0)\,\mathcal{M}\,\hat{\mathcal{K}}_{e1}^{-1} \begin{pmatrix} C\beta \\ 0 \end{pmatrix} \tag{D.25}$$

$$= \langle \mathbf{t}|\frac{T}{1-T^2}|C\beta\rangle - \langle \mathbf{t}|\frac{1}{1-T^2}|C\xi\rangle\langle\xi|\frac{T}{1-T^2}|\beta\rangle - \langle \mathbf{t}|\frac{T}{1-T^2}|\xi\rangle\langle\xi|\frac{1}{1-T^2}|C\beta\rangle$$

$$\mathcal{C}_2 \equiv (0,\mathbf{t})\,\mathcal{M}\,\hat{\mathcal{K}}_{1e}^{-1}\begin{pmatrix}0\\ C\beta\end{pmatrix} \tag{D.26}$$

$$= \langle \mathbf{t}|\frac{T}{1-T^2}|C\beta\rangle - \langle \mathbf{t}|\frac{1}{1-T^2}|\xi\rangle\langle\xi|\frac{T}{1-T^2}|C\beta\rangle - \langle C\mathbf{t}|\frac{T}{1-T^2}|\xi\rangle\langle\xi|\frac{1}{1-T^2}|\beta\rangle$$

and

$$\mathcal{D}_1 \equiv (\mathbf{v}_+ - \mathbf{v}_0, \mathbf{v}_- - \mathbf{v}_+)\,\hat{\mathcal{K}}_{e1}^{-1}\begin{pmatrix}C\beta\\ 0\end{pmatrix}$$

$$= -\langle \mathbf{t}_0|\frac{\rho_1 T+\rho_2}{1-T^2}|C\beta\rangle + \langle \mathbf{t}_0|\frac{1}{1-T^2}|\xi\rangle\left[\langle\xi|\frac{1}{1-T^2}|C\beta\rangle + \langle\xi|\frac{T}{1-T^2}|\beta\rangle\right] \tag{D.27}$$

$$\mathcal{D}_2 \equiv (\mathbf{v}_+ - \mathbf{v}_-, \mathbf{v}_- - \mathbf{v}_0)\,\hat{\mathcal{K}}_{1e}^{-1}\begin{pmatrix}0\\ C\beta\end{pmatrix}$$

$$= -\langle \mathbf{t}_0|\frac{\rho_1 T+\rho_2}{1-T^2}|\beta\rangle + \langle \mathbf{t}_0|\frac{1}{1-T^2}|\xi\rangle\left[\langle\xi|\frac{1}{1-T^2}|\beta\rangle + \langle\xi|\frac{T}{1-T^2}|C\beta\rangle\right] \tag{D.28}$$

D.4 Calculations for the vector state

Applying the formulas of the previous section in the particular case of the vector excitation (5.40) we get

$$|\hat{\varphi}_{e,v}\rangle * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * |\hat{\varphi}_{e,v}\rangle = e^{-\frac{1}{2}G\,p^2}\Big\{d^\mu\langle a^\dagger(1-C)\zeta\rangle$$
$$+ \frac{1}{\kappa+f_e}d^\mu\langle a^\dagger_\mu(1-C)\xi\rangle\,\langle\xi|\frac{f_e+T}{1-T^2}|\zeta\rangle + p\cdot d\Big[-2\langle\mathbf{t}|\frac{T}{1-T^2}|(1-C)\zeta\rangle \tag{D.29}$$
$$+ \langle\mathbf{t}|\frac{T}{1-T^2}|(1-C)\xi\rangle\langle\xi|\frac{1}{1-T^2}|\zeta\rangle + \langle\mathbf{t}|\frac{1}{1-T^2}|(1-C)\xi\rangle\langle\xi|\frac{T}{1-T^2}|\zeta\rangle\Big]\Big\}\mathcal{N}_v|\hat{\varphi}_e(\mathbf{t},p)\rangle$$

A necessary condition to satisfy the LEOM is

$$\langle\xi|\frac{f_e+T}{1-T^2}|\zeta\rangle = 0$$

On the other hand, the presence of the operator $1-C$ in all the terms of the second line tells us that only the $\mathbf{t}_-$ part of $\mathbf{t}$ contributes to this terms. Inserting the explicit form of $\mathbf{t}_-$ one easily finds the result (5.41).

D.5 Level 2 calculations

Using the results of Appendix D, and keeping in mind the formulas

$$\rho_1|0_\pm\rangle = \frac{1}{2}|0_\pm\rangle + \frac{1}{2}|0_\mp\rangle$$
$$\rho_2|0_\pm\rangle = \frac{1}{2}|0_\pm\rangle - \frac{1}{2}|0_\mp\rangle$$

the explicit formulas for the level 2 state are as follows

$$
\begin{aligned}
&\Big(\theta^{\mu\nu}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle|\hat\varphi(\mathbf{t},p)\rangle\Big)*|\hat\Xi\rangle+|\hat\Xi\rangle*\Big(\theta^{\mu\nu}\langle a_\mu^\dagger|\zeta_-\rangle\langle a_\nu^\dagger|\zeta_-\rangle|\hat\varphi(\mathbf{t},p)\rangle\Big) \qquad (D.30)\\
&=e^{-\frac{1}{2}Gp^2}\Bigg[\frac{1}{2}\theta^{\mu\nu}\,\langle a_\mu^\dagger|\zeta_-\rangle\,\langle a_\nu^\dagger|\zeta_-\rangle+2\,\theta_\mu^\mu\,\Bigg(\langle\zeta_-|\frac{T}{1-T^2}|\zeta_-\rangle+2\langle\zeta_-|\frac{1}{1-T^2}|\xi\rangle\langle\xi|\frac{T}{1-T^2}|\zeta_-\rangle\Bigg)\\
&\quad+2\,\theta^{\mu\nu}\,\Big(\langle a_\mu^\dagger|\zeta_+\rangle\,p_\nu\,\mathcal{H}_+ +\langle a_\mu^\dagger|\zeta_-\rangle\,p_\nu\,\mathcal{H}_-\\
&\qquad+\frac{1}{\kappa+1}\langle a_\mu^\dagger(1+C)|\xi\rangle\,\langle\xi|\frac{1}{1-T}|\zeta_-\rangle p_\nu\mathcal{H}_+ +p_\mu p_\nu(\mathcal{H}_+^2+\mathcal{H}_-^2)\Big)\Bigg]\,|\hat\varphi(\mathbf{t},p)\rangle
\end{aligned}
$$

where we have used $|\zeta_+\rangle=(\rho_1-\rho_2)|\zeta_-\rangle$ and we have disregarded terms that explicitly vanish when $\eta\to 0$, i.e. evanescent terms like (5.61). Moreover

$$
\begin{aligned}
\mathcal{H}_+ \;=\;& -\langle\mathbf{t}_+|\frac{T}{1-T^2}|\xi\rangle\langle\xi|\frac{1}{1-T^2}|\zeta_-\rangle+\langle\mathbf{t}_+|\frac{1}{1-T^2}|\xi\rangle\langle\xi|\frac{T}{1-T^2}|\zeta_-\rangle \qquad (D.31)\\
&+\frac{1}{2}\langle\mathbf{t}_0|\frac{1}{1+T}|\zeta_+\rangle+\langle\mathbf{t}_0|\frac{1}{1-T^2}|\xi\rangle\langle\xi|\frac{1}{1+T}|\zeta_-\rangle
\end{aligned}
$$

and

$$
\mathcal{H}_-=-\beta\langle\xi|\frac{T-\kappa}{1-T^2}|\zeta_-\rangle \qquad (D.32)
$$

The other relevant star product is

$$
\begin{aligned}
&\Big(g^\mu\langle a_\mu^\dagger|s_+\rangle|\hat\varphi(\mathbf{t},p)\rangle\Big)*|\hat\Xi\rangle+|\hat\Xi\rangle*\Big(g^\mu\langle a_\mu^\dagger|s_+\rangle|\hat\varphi(\mathbf{t},p)\rangle\Big) \qquad (D.33)\\
&=e^{-\frac{1}{2}Gp^2}\Bigg[g^\mu\langle a_\mu^\dagger|s_+\rangle+\frac{1}{\kappa+1}g^\mu\langle a_\mu^\dagger(1+C)|\xi\rangle\langle\xi|\frac{1}{1-T}|s_+\rangle\\
&\quad-(p\cdot g)\Bigg(\langle\mathbf{t}_0|\frac{1}{1-T}|s_+\rangle-2\,\langle\mathbf{t}_0|\frac{1}{1-T^2}|\xi\rangle\langle\xi|\frac{1}{1-T}|s_+\rangle-2\,\langle\mathbf{t}_+|\frac{T}{1-T^2}|s_+\rangle\\
&\qquad+2\,\langle\mathbf{t}_+|\frac{1}{1-T^2}|\xi\rangle\langle\xi|\frac{T}{1-T^2}|s_+\rangle+2\,\langle\mathbf{t}_+|\frac{T}{1-T^2}|\xi\rangle\langle\xi|\frac{1}{1-T^2}|s_+\rangle\Bigg)\Bigg]\,|\hat\varphi(\mathbf{t},p)\rangle
\end{aligned}
$$

In order for the LEOM to be satisfied the sum of (D.30) and (D.33) must reproduce (5.71). A first condition for this to be true can be easily recognized: the coefficient in front of the $\theta^{\mu\nu}\,\langle a_\mu^\dagger|\zeta_-\rangle\,\langle a_\nu^\dagger|\zeta_-\rangle$ term in the RHS of (D.30) must be 1, which implies $p^2=-1$. This identifies the mass of the solution with the level 2 mass. Next, many terms in the RHS of (D.30,D.33) diverge as $\eta\to 0$. Therefore another condition for LEOM to be satisfied is that the corresponding coefficients vanish. Every bracket in the previous formulas are calculated by going to the k–basis, i.e. by inserting a completeness $\int dk|k\rangle\langle k|$ and then evaluating the k integral. The brackets that contain $|s_+\rangle,|\eta_-\rangle,|\zeta_+\rangle$ involve integrals evaluated essentially at $k=0$; the other brackets are finite. Remembering (5.86), (5.73), (5.74) and moreover that $\mathbf{t}_0$ is finite at $k=0$ (see Appendix A), while $\frac{1}{1+T(k)}\sim 1/k$ and $\xi(k)\to 0$ as $k\to 0$ and $|k_0|>2\eta$, it is easy to determine the degrees of divergence for $\eta\approx 0$. To simplify the analysis we introduce an auxiliary assumption which was already mentioned in the text. We assume that $\xi(k)\neq 0$ only for $k<k_0<0$. This makes all terms containing ξ in the previous formulas irrelevant as far as the LEOM is concerned. Under this hypothesis eq.(D.30) reduces to (5.76) and eq.(D.33) to (5.77). The surviving quantities are as follows

$$\langle\zeta_-|\frac{T}{1-T^2}|\zeta_-\rangle = -\frac{\zeta_0^2 \ln 3}{\pi}\frac{1}{\eta^2} - 2\frac{\zeta_0\zeta_1 \ln 3}{\pi}\frac{1}{\eta} + \pi\zeta_0^2 - \frac{\ln 3}{\pi}(\zeta_1^2 + 2\zeta_0\zeta_2) + \dots \quad \text{(D.34)}$$

$$\mathcal{H}_+ = \frac{1}{2}\langle \mathbf{t}_0|\frac{1}{1+T}|\zeta_+\rangle + \dots = -\frac{\zeta_0 \ln 3}{\sqrt{\pi}}\frac{1}{\eta} - \frac{\zeta_1 \ln 3}{\sqrt{\pi}} + \dots \quad \text{(D.35)}$$

$$\langle \mathbf{t}_0|\frac{1}{1+T}|s_+\rangle = -\frac{2s_{-1}\ln 3}{\sqrt{\pi}}\frac{1}{\eta^2} - \frac{2s_0 \ln 3}{\sqrt{\pi}}\frac{1}{\eta} - (\frac{1}{24}\sqrt{\pi^3}s_{-1} + \frac{2s_1 \ln 3}{\sqrt{\pi}}) + \dots \quad \text{(D.36)}$$

It is important to notice that the numbers (in particular ln3) that appear in this expansion depends heavily on the particular regulator state $|\eta\rangle$ (5.47) we are using. Therefore they should not be attributed any particular significance. This also imply that the conditions we will obtain below are regularization dependent (see comment at the end section 7.1).

Now we can impose the necessary cancelations. We must have

$$2\,\theta^{\mu\nu}\,\langle a_\mu^\dagger|\zeta_+\rangle\, p_\nu\, \mathcal{H}_+ + \frac{1}{2}g^\mu\langle a_\mu^\dagger|s_+\rangle = 0 \quad \text{(D.37)}$$

in the limit $\eta \to 0$. This implies that $g_\mu \sim \theta_{\mu\nu}p^\nu$. Assuming (5.68) we find

$$s_{-1} = -2\sqrt{\frac{2}{\pi}}\, b\, \zeta_0^2 \ln 3\,. \quad \text{(D.38)}$$

The next requirement is that

$$2\,\theta_\mu{}^\mu\langle\zeta_-|\frac{T}{1-T^2}|\zeta_-\rangle + 2\,\theta^{\mu\nu}p_\mu p_\nu \mathcal{H}_+^2 - p\cdot g\,\langle \mathbf{t}_0|\frac{1}{1+T}|s_+\rangle = 0 \quad \text{(D.39)}$$

All three terms diverge like η^{-2} as $\eta \to 0$. The most divergent contribution vanishes if $5ab\ln 3 = 4$. The vanishing of the $1/\eta$ term requires

$$\sqrt{2}\,\zeta_0\zeta_1\,(ab\ln 3 - 4) + \sqrt{\pi}a\, s_0 = 0 \quad \text{(D.40)}$$

This equation binds together the values of s_0, ζ_0, ζ_1. Finally we must impose that also the η^0 term vanishes. This results in an equation of the same type as (D.40), involving also ζ_2 and s_1. It is not very illuminating and therefore we will not write it explicitly.

After imposing these (mild) conditions we see that the linearized EOM is satisfied provided $p^2 = -1$ and the Virasoro constraints in the form (5.68) are satisfied.

To end this appendix, let us add a few lines on how one can do without the auxiliary assumption made before eq.(D.34). In this case we give up this assumption and simply take $\xi(k) \sim k$ as $k \to 0$ (this satisfies (4.16) in a far less restrictive way than the auxiliary condition). Then all the terms in the RHS of (D.30,D.33) are nonvanishing. Two types of terms are dangerous: the term containing $\langle a^\dagger|\zeta_-\rangle$ in the RHS of (D.30) and the two terms proportional to $\langle a^\dagger(1+C)|\xi\rangle$, which are present in both equations. These terms cannot be canceled within the present ansatz for the level 2 state. To deal with the first term we can add to the ansatz (5.71) a term $g^\mu \langle a_\mu^\dagger | r_-\rangle |\hat{\varphi}(\mathbf{t},p)\rangle$ where $|r_-\rangle$ is similar to $|\zeta_-\rangle$, and $r(\eta) = r_0 + r_1\eta + \ldots$. Adjusting the parameter r_0 we can easily cancel the first dangerous term. As for the other two, we can simply add to the ansatz two terms formally equal to the two terms of (5.71), where $|\zeta_-\rangle$ and $|s_+\rangle$ are replaced by $|(1-C)\zeta'\rangle$ and $|(1+C)s'\rangle$, with $\rho_2\zeta' = \zeta', \rho_1\zeta' = 0$ and $\rho_2 r' = r', \rho_1 r' = 0$. We can easily take $\zeta'(k), r'(k)$ to cancel the above two terms as well as all the remaining terms not containing string oscillators $a^\dagger$.

D.6 Level 3 calculations

The first part of this appendix is devoted to redefining the polarizations as mentioned at the beginning of section 7.2. Such redefinitions are as follows

$$
\begin{aligned}
h_\mu &= A\, g_\mu + B\, p\cdot g\, p_\mu \\
\lambda_{\mu\nu} &= C\,\omega_{\mu\nu} + (D_+\, p_\mu \omega_{\rho\nu} + D_-\, p_\nu \omega_{\mu\rho}) p^\rho + D'\, p_\mu p_\nu \omega^{\rho\sigma} p_\rho p_\sigma \\
\chi_{\mu\nu\rho} &= E\,\theta_{\mu\nu\rho} + F\,(p_\mu \theta_{\sigma\nu\rho} + p_\nu \theta_{\mu\sigma\rho} + p_\rho \theta_{\mu\nu\sigma}) p^\sigma \\
&\quad + H\,(p_\mu p_\nu \theta_{\sigma\tau\rho} + p_\mu p_\rho \theta_{\sigma\nu\tau} + p_\nu p_\rho \theta_{\mu\sigma\tau}) p^\sigma p^\tau + H'\, p_\mu p_\nu p_\rho \theta^{\lambda\sigma\tau} p_\lambda p_\sigma p_\tau
\end{aligned}
\tag{D.41}
$$

Inserting the above redefinitions into (5.81,5.82) we get

$$
\begin{aligned}
&3\sqrt{2}\left(\frac{A-2B}{C} - 2\,\frac{A}{C}\,\frac{D_+ + D_- - D'}{D_+ - 2D'}\right) g\cdot p + 2\,\omega_\mu{}^\mu = 0 \\
&3\, g_\mu + \sqrt{2}\frac{C-2D_-}{A}\omega_\mu{}^\nu p_\nu = 0 \\
&2\sqrt{2}\frac{C}{E}\,\omega_{\nu\mu} p^\nu - \sqrt{2}\left(\frac{C+4D_+ - 2D_-}{E} + \frac{(D_+ + D_-)(F-H)}{E(F-2H)}\right)\omega_{\mu\nu} p^\nu + 3\,\theta_{\mu\nu}{}^\nu = 0 \\
&2\,\omega_{(\mu\nu)} + 3\sqrt{2}\,\frac{E-2F}{C}\,\theta_{\mu\nu\rho} p^\rho = 0
\end{aligned}
\tag{D.42}
$$

These equations are of the same form as (5.83,5.84) with an obvious identification of the coefficients x, y, u, v, z. The coefficients $A, \ldots, H'$ are subject to the conditions

$$
\begin{aligned}
&\frac{E-2F}{C} = \frac{H-2H'}{D'}, \qquad \frac{E-2F}{2C} = \frac{F-2H}{D_+ + D_-}, \qquad \frac{C-2D_-}{A} = \frac{D_+ - 2D'}{B} \\
&F\left(C + 2D_- - 4D_+ - \frac{(F-H)(D_+ + D_-)}{F-2H}\right) = E\left(2D_- - D_+ - 2D' - \frac{2D'(H-H')}{H-2H'}\right)
\end{aligned}
$$

The second part of the Appendix concerns the equations that must be verified among the terms of eqs.(5.87,5.88) and (5.89) for the LEOM (5.91) to be satisfied. As explained in the text we have to impose that all the terms in the RHS of eqs.(5.87,5.88) and (5.89) that do not reproduce the level 3 state vanish. There are two such terms: one linear in $a^\dagger$

$$
\begin{aligned}
&3\,\theta_\mu{}^{\mu\rho}\,\langle\zeta_-|\frac{T}{1-T^2}|\zeta_-\rangle\,\langle a_\rho^\dagger|\zeta_-\rangle + 3\,\theta^{\mu\nu\rho}\langle a_\rho^\dagger|\zeta_-\rangle\, p_\mu\, p_\nu\, \mathcal{H}_+^2 \\
&+\omega^{\mu\nu}\left(\langle a_\mu^\dagger|\zeta'_-\rangle p_\nu\,\langle \mathbf{t}_0|\frac{T}{1-T^2}|\lambda_+\rangle \,+\, \langle a_\nu^\dagger|\lambda_-\rangle p_\mu\,\mathcal{H}_+\right) + \frac{3}{4}\,g^\mu\,\langle a_\mu^\dagger|r_-\rangle = 0
\end{aligned}
\tag{D.43}
$$

and another quadratic in $a^\dagger$

$$3\,\theta^{\mu\nu\rho}\,\langle a^\dagger_\mu|\zeta_-\rangle\langle a^\dagger_\nu|\zeta_+\rangle p_\rho \mathcal{H}_+ + \omega^{\mu\nu}\left(\frac{1}{4}\,\langle a^\dagger_\mu|\zeta'_-\rangle\,\langle a^\dagger_\nu|\lambda_+\rangle + \frac{1}{2}\,\langle a^\dagger_\mu|\zeta'_+\rangle\,\langle a^\dagger_\nu|\lambda_-\rangle\right) = 0 \quad \text{(D.44)}$$

Now we use the η–expansions (D.34) and (D.35), together with

$$\langle \mathbf{t}_0|\frac{T}{1-T^2}|\lambda_+\rangle = \frac{\lambda_{-1}\ln 3}{\sqrt{\pi}}\frac{1}{\eta^2} + \left(\frac{\lambda_0 \ln 3}{\sqrt{\pi}} - \frac{\sqrt{\pi}}{4}\lambda_{-1}\right)\frac{1}{\eta} - \left(\frac{\sqrt{\pi^3}}{48}\lambda_{-1} - \frac{\lambda_1 \ln 3}{\sqrt{\pi}} + \frac{\sqrt{\pi}}{4}\lambda_0\right) + \dots$$

Equation (D.44) implies $2\,\omega_{(\mu\nu)} + 3\sqrt{2}\,z\,\theta_{\mu\nu}{}^\rho p_\rho = 0$ for some z. The terms in the RHS of (D.44) are of overall order 0 in η, therefore only one condition is requested:

$$3\,\sqrt{2}\,z\,\zeta'_0\,\lambda_{-1} = 8\,\frac{\zeta_0^3}{\sqrt{\pi}}\ln 3 \quad \text{(D.45)}$$

The RHS of (D.43) contains terms of order –2,–1 and 0 in η as $\eta \to 0$. We must therefore satisfy three conditions. Using that $\theta^{\mu\nu\rho}p_\mu p_\nu \sim \omega^{(\rho\mu)}p_\mu$, we see that the condition involving the term of order –2, takes exactly the form of the first equation (5.84) with

$$u = \sqrt{\frac{\pi}{2}}\,\frac{\zeta'_0\lambda_{-1}}{\zeta_0^3} - \frac{\ln 3}{6z} \quad \text{(D.46)}$$

$$v = \frac{1}{2}\sqrt{\frac{\pi}{2}}\,\frac{\lambda_1}{\zeta_0^2} + \frac{\ln 3}{12z} \quad \text{(D.47)}$$

For generic values of $\zeta_0, \zeta'_0, \lambda_{-1}$, eqs.(D.45,D.46,D.47) fix u, v and z to some specific (non-vanishing) values. Now the vanishing of the term $\sim \eta^{-1}$ leads to an equation similar to the first equation (5.84), with identifications for u and v different from (D.46,D.47),

$$u = -\frac{2}{3}\sqrt{\frac{\pi}{2}}\,\frac{\zeta'_0\lambda_0 + \zeta'_1\lambda_{-1}}{\zeta_0^2\zeta_1} - \frac{\ln 3}{6z} \quad \text{(D.48)}$$

$$v = \frac{1}{6\sqrt{2}}\,\frac{\lambda_1\zeta_1 + \lambda_0\zeta_0}{\zeta_1\zeta_0^2} + \frac{\ln 3}{12z} \quad \text{(D.49)}$$

These equations however involve three additional parameters $\zeta_1, \lambda_0, \zeta'_1$. So it is easy to tune them to the same specific values for u and v. Finally the term of order η^0 involves also g^μ. In this case there are several possibilities[1]. One of these is that $g_\mu \sim \omega_{\nu\mu}p^\nu$. In the latter case also the constant y in (5.83) gets determined in terms of all the parameters, which include now also $\zeta_2, \zeta'_2, \lambda_1, r_0$. Since the relevant equations are cumbersome and not particularly illuminating we do not write them down. In conclusion the LEOM for the state (5.85) is satisfied together with the Virasoro constraints (5.83,5.84) (the first is a consequence of the other three), provided some mild conditions on the various parameters that enter the game be complied with.

[1] In order to restrict the number of these possibilities and obtain more binding conditions one should give up the simplifying assumption and treat the level 3 in full generality.

D.7 The cochain space

In this Appendix we would like to explain in more detail the definition of the space of cochains given in section 8.

From eq.(5.95), it would seem that, should we keep η finite throughout the cohomological analysis, all the states we have constructed would be trivial. This is due to the fact that in the gauge transformed expressions there appears the operator $\rho_1 - \rho_2$, which has the property that $(\rho_1 - \rho_2)|\eta_\pm\rangle = |\eta_\mp\rangle$. However this would be misleading, since in this argument we forget all the corrections to the LEOM that vanish only when $\eta \to 0$. In addition one should not forget that the η dependence is an artifact of our regularization, it does not correspond to anything that has to do with the physical string modes. It can only appear at an intermediate step in our calculations. Therefore the space of cochains should not contain any reference to the η dependence. There are only two ways to implement this. We can say that every cochain is defined up to evanescent states. But this would lead to incurable inconsistencies: for instance, the 0 state would be defined up to evanescent states, but we know that by applying, for instance, a gauge transformation to $|\eta_+\rangle$, which is evanescent, we get $|\eta_-\rangle$, which is in a nonzero class; so we would get the paradoxical result that applying the BRST operator to 0, we get something different from zero. This possibility has consequently to be excluded.

The only consistent possibility is the one put forward in the text. The nonzero cochains are those obtained by explicitly taking the limit $\eta \to 0$ for non-evanescent states, that is taking the limit in expressions of the type $\langle a^\dagger \zeta\rangle$ both for regular and singular ζ's (see section 6, in particular formulas (5.51)). It is clear that one gets in this way well–defined expressions for the states. This will form the set of nonzero cochains. To this we have to add the zero cochain, which is simply the zero state. All together they form a linear space. By definition this is the space of cochains where we want to compute the cohomology of the VSFT fluctuations. The η–regularization enters into the game when we come to compute the star products of the LEOM or of (5.93). Without such regularization these star products are ambiguous. From this point of view we see that the η–regularization concerns the star product rather than the states themselves. The cohomological problem at this point is well–defined.

D.8 Towers of solutions

In this appendix we prove the statements used in section 8 to show that for any solutions to the LEOM we can construct an infinite tower of solutions with the same mass. We begin with the calculation of the star product $(h^\nu\langle a_\nu^\dagger \xi\rangle|\hat{\varphi}(\theta, n, \mathbf{t}, p)\rangle) * |\hat{\Xi}\rangle$. Written down explicitly this becomes

$$\left(h^\nu\langle a_\nu^\dagger\xi\rangle\sum_{i=1}^n\theta_i^{\mu_1\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat\varphi(\mathbf{t},p)\rangle\right)*|\hat\Xi\rangle \;=\; h^\nu\sum_{i=1}^n\theta_i^{\mu_1\ldots\mu_i}\times \tag{D.50}$$
$$\times\langle-\xi\frac{\partial}{\partial\beta^\nu}\rangle\langle-\zeta_1^{(i)}\frac{\partial}{\partial\beta^{\mu_1}}\rangle\ldots\langle-\zeta_i^{(i)}\frac{\partial}{\partial\beta^{\mu_i}}\rangle\exp\left[-\frac{1}{2}\mathcal{A}_1-\mathcal{B}_1-p\,(\mathcal{C}_1+\mathcal{D}_1)\right]|\hat\varphi(\mathbf{t},p)\rangle\Big|_{\beta=0}$$

Now we set $\mathcal{F}_1=-\frac{1}{2}\mathcal{A}_1$ and $\mathcal{G}_1=-\mathcal{B}_1-p\,(\mathcal{C}_1+\mathcal{D}_1)$. Then the RHS of (D.50) becomes

$$\begin{aligned}
&= h^\nu\sum_{i=1}^n\theta_i^{\mu_1\ldots\mu_i}\langle-\zeta_1^{(i)}\frac{\partial}{\partial\beta^{\mu_1}}\rangle\ldots\langle-\zeta_i^{(i)}\frac{\partial}{\partial\beta^{\mu_i}}\rangle\langle\xi\frac{\partial(\mathcal{F}_1+\mathcal{G}_1)}{\partial\beta^\nu}\rangle\,\mathrm{e}^{[\mathcal{F}_1+\mathcal{G}_1]}\Big|_{\beta=0}|\hat\varphi(\mathbf{t},p)\rangle\\
&= h^\nu\sum_{i=1}^n\theta_i^{\mu_1\ldots\mu_i}\langle-\zeta_1^{(i)}\frac{\partial}{\partial\beta^{\mu_1}}\rangle\ldots\langle-\zeta_i^{(i)}\frac{\partial}{\partial\beta^{\mu_i}}\rangle\langle-\xi\frac{\partial\mathcal{G}_1}{\partial\beta^\nu}\rangle\,\mathrm{e}^{[\mathcal{F}_1+\mathcal{G}_1]}\Big|_{\beta=0}|\hat\varphi(\mathbf{t},p)\rangle+\\
&+h^\nu\sum_{i=1}^n\theta_i^{\mu_1\ldots\mu_i}\sum_{j=1}^i\left(\langle\xi\frac{\partial^2\mathcal{F}_1}{\partial\beta^\nu\,\partial\beta^{\mu_j}}\zeta_j^{(i)}\rangle\langle-\zeta_1^{(i)}\frac{\partial}{\partial\beta^{\mu_1}}\rangle\ldots\widetilde{\langle-\zeta_j^{(i)}\frac{\partial}{\partial\beta^{\mu_j}}\rangle}\ldots\langle-\zeta_i^{(i)}\frac{\partial}{\partial\beta^{\mu_i}}\rangle\right)\times\\
&\quad\times\mathrm{e}^{[\mathcal{F}_1+\mathcal{G}_1]}\Big|_{\beta=0}|\hat\varphi(\mathbf{t},p)\rangle\\
&= h^\nu\langle-\xi\frac{\partial\mathcal{G}_1}{\partial\beta_\nu}\rangle\left(\sum_{i=1}^n\theta_i^{\mu_1\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat\varphi(\mathbf{t},p)\rangle*|\hat\Xi\rangle\right)\\
&+h^\nu\sum_{j=1}^i\langle\xi\frac{\partial^2\mathcal{F}_1}{\partial\beta^\nu\,\partial\beta^{\mu_j}}\zeta_j^{(i)}\rangle\left(\theta_i^{\mu_1\ldots\mu_j\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\widetilde{\langle a_{\mu_j}^\dagger\zeta_j^{(i)}\rangle}\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat\varphi(\mathbf{t},p)\rangle*|\hat\Xi\rangle\right)
\end{aligned}$$

Tilded quantities denote omitted ones. Now, using the formulas of Appendix D, and the fact that $\rho_2\xi=\xi,\rho_1\xi=0$, it is easy to prove that

$$\langle-\xi\frac{\partial\mathcal{G}_1}{\partial\beta_\nu}\rangle=\langle a_\nu^\dagger\xi\rangle,\qquad\langle\xi\frac{\partial^2\mathcal{F}_1}{\partial\beta^\nu\,\partial\beta^{\mu_j}}\zeta_j^{(i)}\rangle=\eta_{\nu\mu_j}\langle\xi|\frac{\kappa-T}{1-T^2}|C\zeta_j^{(i)}\rangle$$

Inserting this back in the previous equations, we obtain

$$\begin{aligned}
&\left(h^\nu\langle a_\nu^\dagger\xi\rangle\sum_{i=1}^n\theta_i^{\mu_1\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat\varphi(\mathbf{t},p)\rangle\right)*|\hat\Xi\rangle=\\
&=\;h^\nu\langle a_\nu^\dagger\xi\rangle\left[\sum_{i=1}^n\theta_i^{\mu_1\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat\varphi(\mathbf{t},p)\rangle*|\hat\Xi\rangle\right]\\
&+h^\nu\sum_{j=1}^i\eta_{\nu\mu_j}\langle\xi|\frac{\kappa-T}{1-T^2}|C\zeta_j^{(i)}\rangle\left(\theta_i^{\mu_1\ldots\mu_j\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\widetilde{\langle a_{\mu_j}^\dagger\zeta_j^{(i)}\rangle}\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat\varphi(\mathbf{t},p)\rangle*|\hat\Xi\rangle\right)
\end{aligned}\tag{D.51}$$

Now we repeat the calculation for the commuted product

$$|\hat\Xi\rangle*\left(h^\nu\langle a_\nu^\dagger\xi\rangle\sum_{i=1}^n\theta_i^{\mu_1\ldots\mu_i}\langle a_{\mu_1}^\dagger\zeta_1^{(i)}\rangle\ldots\langle a_{\mu_i}^\dagger\zeta_i^{(i)}\rangle|\hat\varphi(\mathbf{t},p)\rangle\right)\;=\;h^\nu\sum_{i=1}^n\theta_i^{\mu_1\ldots\mu_i}\times \tag{D.52}$$
$$\times\langle-\xi\frac{\partial}{\partial\beta^\nu}\rangle\langle-\zeta_1^{(i)}\frac{\partial}{\partial\beta^{\mu_1}}\rangle\ldots\langle-\zeta_i^{(i)}\frac{\partial}{\partial\beta^{\mu_i}}\rangle\exp\left[-\frac{1}{2}\mathcal{A}_2-\mathcal{B}_2-p\,(\mathcal{C}_2+\mathcal{D}_2)\right]|\hat\varphi(\mathbf{t},p)\rangle\Big|_{\beta=0}$$

Now, to simplify notation, we set $\mathcal{F}_2 = -\frac{1}{2}\mathcal{A}_2$ and $\mathcal{G}_2 = -\mathcal{B}_2 - p\,(\mathcal{C}_2 + \mathcal{D}_2)$, and proceeding as before (D.52) becomes

$$
\begin{aligned}
&= h^\nu \langle -\xi \frac{\partial \mathcal{G}_2}{\partial \beta_\nu}\rangle \left(|\hat{\Xi}\rangle * \left(\sum_{i=1}^{n} \theta_i^{\mu_1 \ldots \mu_i} \langle a_{\mu_1}^\dagger \zeta_1^{(i)}\rangle \ldots \langle a_{\mu_i}^\dagger \zeta_i^{(i)}\rangle |\hat{\varphi}(\mathbf{t},p)\rangle \right)\right) \\
&+ h^\nu \sum_{j=1}^{i} \langle \xi \frac{\partial^2 \mathcal{F}_2}{\partial \beta^\nu \, \partial \beta^{\mu_j}} \zeta_j^{(i)}\rangle \left(|\hat{\Xi}\rangle * \left(\theta_i^{\mu_1 \ldots \mu_j \ldots \mu_i} \langle a_{\mu_1}^\dagger \zeta_1^{(i)}\rangle \ldots \widetilde{\langle a_{\mu_j}^\dagger \zeta_j^{(i)}\rangle} \ldots \langle a_{\mu_i}^\dagger \zeta_i^{(i)}\rangle |\hat{\varphi}(\mathbf{t},p)\rangle \right)\right) \\
&= h^\nu \langle a_\nu^\dagger \xi\rangle \left[|\hat{\Xi}\rangle * \left(\sum_{i=1}^{n} \theta_i^{\mu_1 \ldots \mu_i} \langle a_{\mu_1}^\dagger \zeta_1^{(i)}\rangle \ldots \langle a_{\mu_i}^\dagger \zeta_i^{(i)}\rangle |\hat{\varphi}(\mathbf{t},p)\rangle \right)\right] \qquad \text{(D.53)}
\end{aligned}
$$

since

$$
\langle -\xi \frac{\partial \mathcal{G}_2}{\partial \beta_\nu}\rangle = \langle a_\nu^\dagger \xi\rangle, \qquad \langle \xi \frac{\partial^2 \mathcal{F}_2}{\partial \beta^\nu \, \partial \beta^{\mu_j}} \zeta_j^{(i)}\rangle = 0
$$

Collecting the above results we have

$$
\begin{aligned}
&\left(h^\nu \langle a_\nu^\dagger \xi\rangle |\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle \right) * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * \left(h^\nu \langle a_\nu^\dagger \xi\rangle |\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle \right) = \\
&= \; h^\nu \langle a_\nu^\dagger \xi\rangle \left[|\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * |\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle \right] + \qquad \text{(D.54)} \\
&+ h^\nu \sum_{j=1}^{i} \eta_{\nu\mu_j} \langle \xi | \frac{\kappa - T}{1 - T^2} | C \zeta_j^{(i)}\rangle \left(\theta_i^{\mu_1 \ldots \mu_j \ldots \mu_i} \langle a_{\mu_1}^\dagger \zeta_1^{(i)}\rangle \ldots \widetilde{\langle a_{\mu_j}^\dagger \zeta_j^{(i)}\rangle} \ldots \langle a_{\mu_i}^\dagger \zeta_i^{(i)}\rangle |\hat{\varphi}(\mathbf{t},p)\rangle * |\hat{\Xi}\rangle \right)
\end{aligned}
$$

The last line vanishes if $\langle \xi | \frac{T-\kappa}{1-T^2} | C \zeta_j^{(i)}\rangle = 0$ or if, for those j's for which this is not true, h is transverse to the tensor θ. In this case, if $|\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle$ is a solution to the linearized equation of motion,

$$
\left(h^\nu \langle a_\nu^\dagger \xi\rangle |\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle \right) * |\hat{\Xi}\rangle + |\hat{\Xi}\rangle * \left(h^\nu \langle a_\nu^\dagger \xi\rangle |\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle \right) = h^\nu \langle a_\nu^\dagger \xi\rangle |\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle \qquad \text{(D.55)}
$$

i.e. also $h^\nu \langle a_\nu^\dagger \xi\rangle |\hat{\varphi}(\theta,n,\mathbf{t},p)\rangle$ is a solution. All the results similar to this used in section 8 can be obtained by obvious extensions of the previous argument.

D.9 Calculating H

The number H comes from the three–point tachyon vertex. If we take (5.38) as the tachyon solution, the three–tachyon vertex is given by

$$
{}_1\langle \phi_e(\mathbf{t},p_1)|_2\langle \phi_e(\mathbf{t},p_2)|_3\langle \phi_e(\mathbf{t},p_3)|V_3\rangle_{123} \; = \; \left(\det \hat{\mathcal{K}}_3\right)^{-\frac{D}{2}} \hat{\mathcal{N}}_e^3 \exp[-\mathcal{H}_1(p_1,p_2,p_3)] \qquad \text{(D.56)}
$$

$\mathcal{H}_1$ is given by

$$
\mathcal{H}_1(p_1,p_2,p_3) = \chi^T \hat{\mathcal{K}}_3^{-1} \lambda + \frac{1}{2} \lambda^T \mathcal{V}_3 \hat{\mathcal{K}}_3^{-1} \lambda + \frac{1}{2} \chi^T \hat{\mathcal{K}}_3^{-1} \hat{\Sigma}_3 \chi + \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) V_{00} \qquad \text{(D.57)}
$$

with $p_1+p_2+p_3=0$. In this equation the various symbols are as follows

$$\begin{aligned}
&\lambda^T=(\lambda_1,\lambda_2,\lambda_3), \qquad \lambda_i=-p_i\mathbf{t}C, \quad i=1,2,3\\
&\chi=\begin{pmatrix}\mathbf{v}_0p_1+\mathbf{v}_-p_2+\mathbf{v}_+p_3\\ \mathbf{v}_+p_1+\mathbf{v}_0p_2+\mathbf{v}_-p_3\\ \mathbf{v}_-p_1+\mathbf{v}_+p_2+v_0p_3\end{pmatrix}=\begin{pmatrix}\mathbf{v}_0 & \mathbf{v}_- & \mathbf{v}_+\\ \mathbf{v}_+ & \mathbf{v}_0 & \mathbf{v}_-\\ \mathbf{v}_- & \mathbf{v}_+ & \mathbf{v}_0\end{pmatrix}\begin{pmatrix}p_1\\ p_2\\ p_3\end{pmatrix} \qquad \text{(D.58)}\\
&\hat\Sigma_3=\begin{pmatrix}\hat S_e & 0 & 0\\ 0 & \hat S_e & 0\\ 0 & 0 & \hat S_e\end{pmatrix}, \qquad \mathcal{V}_3=\begin{pmatrix}V & V_+ & V_-\\ V_- & V & V_+\\ V_+ & V_- & V\end{pmatrix}
\end{aligned}$$

Finally $\hat{\mathcal{K}}_3=1-\hat\Sigma_3\mathcal{V}_3$. Since

$$\mathcal{N}_e = \frac{g_0}{\sqrt{G}}\sqrt{\frac{\det(1-\hat S_e^2)^{\frac{D}{2}}}{\det(1-\hat{\hat S}^2)}}\exp[-\frac{1}{2}p^2\mathbf{t}\frac{1}{1+\hat T_e}C\mathbf{t}] \qquad \text{(D.59)}$$

the total exponential in (D.56) is given by

$$\mathcal{H}=\mathcal{H}_1+\mathcal{H}_2, \qquad \mathcal{H}_2(p_1,p_2,p_3)=(p_1^2+p_2^2+p_3^2)H_2, \qquad H_2=-\frac{1}{2}\langle\mathbf{t}|\frac{1}{1+\hat T_e}C|\mathbf{t}\rangle \quad \text{(D.60)}$$

Similarly one can show that $\mathcal{H}_1(p_1,p_2,p_3)=(p_1^2+p_2^2+p_3^2)H_1$. Let us set $H=H_1+H_2$.

All expressions can be straightforwardly computed once we explicitly determine the quantity

$$\hat{\mathcal{K}}_3^{-1}=\left(1-\hat\Sigma_3\mathcal{V}\right)^{-1}$$

It turns out that

$$\hat{\mathcal{K}}_3^{-1}=\mathcal{K}_3^{-1}\left[1+\left(1-\mathcal{P}_e\mathcal{M}_3\mathcal{K}_3^{-1}\right)^{-1}\mathcal{P}_e\mathcal{M}_3\mathcal{K}_3^{-1}\right] \qquad \text{(D.61)}$$

Moreover we have

$$\begin{aligned}
\left(1-\mathcal{P}_e\mathcal{M}_3\mathcal{K}_3^{-1}\right)^{-1}\mathcal{P}_e &= \sum_{n=0}^{\infty}\left(\mathcal{P}_e\mathcal{M}_3\mathcal{K}_3^{-1}\right)^n\mathcal{P}_e\\
&=\frac{\kappa+f_e}{f_e^3-1}\begin{pmatrix} f_e^2 & (f_e\rho_1+\rho_2) & (f_e\rho_2+\rho_1)\\ (f_e\rho_2+\rho_1) & f_e^2 & (f_e\rho_1+\rho_2)\\ (f_e\rho_1+\rho_2) & (f_e\rho_2+\rho_1) & f_e^2\end{pmatrix}\mathcal{P}_e \qquad \text{(D.62)}
\end{aligned}$$

With the use of this formula one can directly compute all the contributions in (D.57) given the general tachyon solution

$$\begin{aligned}
\mathbf{t} &= \mathbf{t}_+ + \mathbf{t}_-\\
\mathbf{t}_+ &= \mathbf{t}_0+\alpha\, W\,(\xi+C\xi)\,, \quad W=\langle\xi|\frac{1}{1+T}|\mathbf{t}_0\rangle \qquad \text{(D.63)}\\
\mathbf{t}_- &= \beta\,(\xi-C\xi)
\end{aligned}$$

When momentum conservation holds we have the following identity

$$(p_1,p_2,p_3)\begin{pmatrix}a & b & c\\ c & a & b\\ b & c & a\end{pmatrix}\begin{pmatrix}p_1\\ p_2\\ p_3\end{pmatrix}\Bigg|_{\sum_i p_i=0}=\left(a-\frac{1}{2}(b+c)\right)\sum_i p_i^2 \qquad \text{(D.64)}$$

Let us begin analyzing the contribution coming from the twist–even part of tachyon. With lenghty but straightforward manipulations we get

$$\chi^T\hat{\mathcal{K}}_3^{-1}\lambda = -\frac{1}{2}(p_1^2+p_2^2+p_3^2)\Big[\langle\mathbf{t}_0|\frac{1}{1-T^2}|\mathbf{t}_+\rangle - \frac{2}{f_e^2+f_e+1}\langle\mathbf{t}_0|\frac{1}{1-T^2}|\xi\rangle\left((f_e-1)\langle\xi|\frac{T}{1-T^2}|\mathbf{t}_+\rangle-(1+2f_e)\langle\xi|\frac{1}{1-T^2}|\mathbf{t}_+\rangle\right)\Big] \tag{D.65}$$

$$\begin{aligned}\lambda^T\mathcal{V}_3\hat{\mathcal{K}}_3^{-1}\lambda &= (p_1^2+p_2^2+p_3^2)\Big[\langle\mathbf{t}_+|\frac{T}{1-T^2}|\mathbf{t}_+\rangle-\frac{1}{2}\langle\mathbf{t}_+|\frac{1}{1-T^2}|\mathbf{t}_+\rangle\\ &+\frac{1}{f_e^2+f_e+1}\Big((f_e-1)\langle\mathbf{t}_+|\frac{1}{1-T^2}|\xi\rangle^2-(2f_e+1)\langle\mathbf{t}_+|\frac{T}{1-T^2}|\xi\rangle^2\\ &+2(f_e+2)\langle\mathbf{t}_+|\frac{1}{1-T^2}|\xi\rangle\langle\mathbf{t}_+|\frac{T}{1-T^2}|\xi\rangle\Big)\Big]\end{aligned} \tag{D.66}$$

$$\chi^T\hat{\mathcal{K}}_3^{-1}\hat{\Sigma}_3\chi = \frac{3}{2}(p_1^2+p_2^2+p_3^2)\Big(\langle\mathbf{t}_0|\frac{T(1-2T)}{(1-T^2)(1+T)}|\mathbf{v}_0\rangle+\frac{f_e+2}{f_e^2+f_e+1}\langle\mathbf{t}_0|\frac{1}{1-T^2}|\xi\rangle^2\Big) \tag{D.67}$$

Plugging inside the expression for $\mathbf{t}_+$ we get

$$\begin{aligned}\mathcal{H}_1^+(p_1,p_2,p_3) &= (p_1^2+p_2^2+p_3^2)\Big\{\frac{1}{2}\langle\mathbf{t}_0|\frac{1}{1+T}|\mathbf{t}_0\rangle+2\alpha W^2+\alpha^2(\kappa-\frac{1}{2})W^2\\ &-\frac{1}{f_e^2+f_e+1}\,\frac{1}{2}\Big[\alpha^2\left(f_e(1+2\kappa-2\kappa^2)-(1-4\kappa+\kappa^2)\right)\\ &+2\alpha\left(f_e(2\kappa-1)+(\kappa-2)\right)-(2f_e+1)\Big]W^2\Big\}\end{aligned} \tag{D.68}$$

The second contribution to H comes from the normalization in front of the tachyon state (D.59), that is

$$\mathcal{H}_2^+(p_1,p_2,p_3) = -\frac{1}{2}\,(p_1^2+p_2^2+p_3^2)\,\langle\mathbf{t}_+|\frac{1}{1+\hat{T}_e}C|\mathbf{t}_+\rangle$$

We have

$$\langle\mathbf{t}_+|\frac{1}{1+\hat{T}_e}C|\mathbf{t}_+\rangle = \langle\mathbf{t}_+|\frac{1}{1+T}|\mathbf{t}_+\rangle+\frac{2}{\kappa+f_e}\langle\mathbf{t}_+|\frac{1}{1+T}|\xi\rangle^2\sum_{n=0}^{\infty}\left(\frac{\kappa-1}{\kappa+f_e}\right)^n \tag{D.69}$$

$$= \langle\mathbf{t}_0|\frac{1}{1+T}|\mathbf{t}_0\rangle+4\alpha W^2+2\alpha^2(\kappa-1)W^2+\frac{2}{f_e+1}\left(\alpha(\kappa-1)+1\right)^2W^2 \tag{D.70}$$

The total twist–even contribution in H, let us call it H^+, is then

$$H^+ = H_1^+ + H_2^+ \tag{D.71}$$

$$= H_0-\frac{(f_e-1)^2(\kappa+f_e)^2}{2(f_e+1)(f_e^3-1)}\left(\frac{1}{\kappa+f_e}-\alpha\right)^2\langle\mathbf{t}_0|\frac{1}{1+T}|\xi\rangle^2 \tag{D.72}$$

The bare contribution H_0 is naively zero but, in level truncation regularization, it acquires a non–vanishing value, [62]. We stress once more that the dressing contribution is not affected by twist anomaly as the half string vector ξ does not excite the $k=0$ (zero momentum) midpoint mode.

Now we turn to the twist–odd contributions which, for $e \neq 1$, does not vanish identically for any solution to the LEOM. Let's analyze first the purely imaginary contribution linear in β. It is easy to see that the part coming from H_2 is identically zero by twist symmetry, and the same is true for the term $\lambda_-^T \mathcal{V}_3 \hat{\mathcal{K}}_3^{-1} \lambda_+$ in H_1. So the only potential contributions arise from the tachyon linear term $\chi^T \hat{\mathcal{K}}_3^{-1} \lambda_-$. It is straightforward to compute these terms by plugging $\mathbf{t}_- = \beta\,(\xi - C\xi)$ and to show again that twist symmetry requires this contribution to vanish. So there are no imaginary contributions in H.
The quadratic terms in β come out from $\lambda_-^T \mathcal{V}_3 \hat{\mathcal{K}}_3^{-1} \lambda_-$ in H_1 and $\langle \mathbf{t}_- | \frac{1}{1+\hat{T}_e} C | \mathbf{t}_- \rangle$ in H_2. They can be directly computed plugging the explicit expression for $\mathbf{t}_-$. The result is

$$\lambda_-^T \mathcal{V}_3 \hat{\mathcal{K}}_3^{-1} \lambda_- = \beta^2 \frac{(f_e + \kappa)(f_e(2\kappa - 1) + (\kappa - 2))}{f_e^2 + f_e + 1} \sum_i p_i^2 \tag{D.73}$$

$$-\frac{1}{2} \langle \mathbf{t}_- | \frac{1}{1+\hat{T}_e} C | \mathbf{t}_- \rangle = -\beta^2 \frac{(\kappa - 1)(\kappa + f_e)}{f_e + 1} \tag{D.74}$$

Together they sum up to

$$\mathcal{H}^- \equiv H^- \sum_i p_i^2 = \frac{1}{2} \lambda_-^T \mathcal{V}_3 \hat{\mathcal{K}}_3^{-1} \lambda_- - \frac{1}{2} \langle \mathbf{t}_- | \frac{1}{1+\hat{T}_e} C | \mathbf{t}_- \rangle \sum_i p_i^2 \tag{D.75}$$

$$H^- = \beta^2 \frac{(f_e - 1)^2 (\kappa + f_e)^2}{2(f_e + 1)(f_e^3 - 1)} \tag{D.76}$$

Appendix E

Spectroscopy of Neumann matrices with B field

In this appendix we present the computation of the eigenstates and eigenvalues of the Neumann matrix ${\mathcal{X}^\alpha}_\beta$ in the presence of B–field, along the line of [50]. A similar analysis was carried out in [79], but with no reference to the correct normalization of continuous and discrete eigenvectors; moreover the discrete eigenvectors presented in the first of [79] does not reproduce the known ones when $B \to 0$. Since the discrete spectrum is of crucial importance for our purposes we re–derive completely the whole spectroscopy taking care of the correct normalization of continuous and discrete eigenvalues, as in [50]. To avoid the degeneracy of the diagonal Neumann coefficient ${\mathcal{X}^\alpha}_\beta$, we consider the unitary matrices $C'{\mathcal{U}^\alpha}_\beta$ and ${\mathcal{U}^\alpha}_\beta C'$, which are related to ${\mathcal{X}^\alpha}_\beta$ as follows [66, 63, 67]

$$\left({\mathcal{X}^\alpha}_\beta\right)_{NM} = \frac{1}{3}\left({\delta^\alpha}_\beta + C'{\mathcal{U}^\alpha}_\beta + C'{\bar{\mathcal{U}}^\alpha}_\beta\right)_{NM}. \tag{E.1}$$

The matrix $(C'{\mathcal{U}^\alpha}_\beta)_{NM}$ can be written explicitly as

$$C'\mathcal{U} \;=\; \left(\begin{array}{cccc} 1-3bK & 2\sqrt{3}bKa & 3\sqrt{b}K\langle W| & -2\sqrt{3b}Ka\langle W| \\ -2\sqrt{3}bKa & 1-3bK & 2\sqrt{3b}Ka\langle W| & 3\sqrt{b}K\langle W| \\ 3\sqrt{b}K|W\rangle & -2\sqrt{3b}Ka|W\rangle & CU-3K|W\rangle\langle W| & 2\sqrt{3}Ka|W\rangle\langle W| \\ 2\sqrt{3b}Ka|W\rangle & 3\sqrt{b}K|W\rangle & -2\sqrt{3}Ka|W\rangle\langle W| & CU-3K|W\rangle\langle W| \end{array}\right)$$

where, see [5]

$$|W\rangle = -\sqrt{2}(|v_e\rangle + i|v_o\rangle), \qquad\qquad K = \frac{A^{-1}}{4a^2+3}. \tag{E.2}$$

CU is the non-zero mode analog of $C'\mathcal{U}$ without B field. We recall that, [66, 63, 67],

$$C'\bar{\mathcal{U}} = \tilde{\mathcal{U}}C', \tag{E.3}$$

where tilde means transposition with respect to α, β indices.

Our aim is to solve the eigenvalue equation

$$C'\mathcal{U}|\Psi\rangle = \xi|\Psi\rangle, \qquad |\Psi\rangle = \begin{pmatrix} g_1 \\ g_2 \\ |\Lambda_1\rangle \\ |\Lambda_2\rangle \end{pmatrix}, \tag{E.4}$$

which splits into

$$\langle W|\Lambda_1\rangle = \frac{A}{\sqrt{b}}[\xi - 1 + \frac{b}{A}]g_1 + \frac{2Aa}{\sqrt{3b}}(\xi - 1)g_2 \tag{E.5}$$

$$\langle W|\Lambda_2\rangle = \frac{A}{\sqrt{b}}[\xi - 1 + \frac{b}{A}]g_2 - \frac{2Aa}{\sqrt{3b}}(\xi - 1)g_1 \tag{E.6}$$

$$(CU - \xi)|\Lambda_1\rangle = \sqrt{\frac{1}{b}}g_1(\xi - 1)|W\rangle \tag{E.7}$$

$$(CU - \xi)|\Lambda_2\rangle = \sqrt{\frac{1}{b}}g_2(\xi - 1)|W\rangle \tag{E.8}$$

We know, [28], that CU has a continuous spectrum and the solution of (E.4) depends on whether the eigenvalue ξ is in the continuous spectrum of CU or not. So we will distinguish these two different cases and analyze each of them in detail.

E.1 Discrete spectrum

If ξ is not in the spectrum of CU, we can invert $(CU - \xi)$ in equations (E.7) and (E.8) to obtain

$$|\Lambda_1\rangle = \sqrt{\frac{1}{b}}g_1(\xi - 1)\frac{1}{(CU - \xi)}|W\rangle \tag{E.9}$$

$$|\Lambda_2\rangle = \sqrt{\frac{1}{b}}g_2(\xi - 1)\frac{1}{(CU - \xi)}|W\rangle. \tag{E.10}$$

As we can see the solutions get modified w.r.t. the $B = 0$ case, only via possible modifications of the eigenvalue ξ. Substitution of these solutions into equations (E.5) and (E.6) gives

$$\sqrt{\frac{1}{2b}}(\xi - 1)\langle W|\frac{1}{CU - \xi}|W\rangle g_1 - \frac{A}{\sqrt{2b}}\left(\xi - 1 + \frac{b}{A}\right)g_1 - \frac{2aA}{\sqrt{6b}}(\xi - 1)g_2 = 0 \tag{E.11}$$

$$\sqrt{\frac{1}{2b}}(\xi - 1)\langle W|\frac{1}{CU - \xi}|W\rangle g_2 - \frac{A}{\sqrt{2b}}\left(\xi - 1 + \frac{b}{A}\right)g_2 + \frac{2aA}{\sqrt{6b}}(\xi - 1)g_1 = 0 \tag{E.12}$$

The bracket which appears here is the same as the one in [50] and is given by

$$\langle W|\frac{1}{CU-\xi}|W\rangle = V_{00} + \frac{\xi+1}{\xi-1}2\Re F(\eta) \tag{E.13}$$

where

$$F(\eta) = \psi\left(\frac{1}{2}+\frac{\eta}{2\pi i}\right) - \psi\left(\frac{1}{2}\right), \qquad \xi = -\frac{1}{1-2\mathrm{cosh}\eta}[2-\mathrm{cosh}\eta - \mathrm{i}\sqrt{3}\mathrm{sinh}\eta]. \tag{E.14}$$

$\psi(x)$ is the logarithmic derivative of the Euler Γ-function.

Substitution of these in (E.11) and (E.12) gives us

$$\left(\Re F(\eta) - \frac{b}{4}\right)g_1 - \frac{aA}{\sqrt{3}}\frac{\xi-1}{\xi+1}g_2 = 0, \qquad \left(\Re F(\eta) - \frac{b}{4}\right)g_2 + \frac{aA}{\sqrt{3}}\frac{\xi-1}{\xi+1}g_1 = 0. \tag{E.15}$$

This system of equations will have non trivial solutions for g_2 and g_1 if the determinant of the coefficient matrix is zero, i.e.

$$\frac{b}{4} = \Re F(\eta) \pm aA\mathrm{tanh}\frac{\eta}{2}, \tag{E.16}$$

Using equations (E.15) we can show that $g_2 = \pm i g_1$. This is a constraint on g_1 and g_2 thus we cannot split the eigenstates in the two directions, choosing one of the constants to be zero. g_1 is now an overall constant, which can be chosen real and fixed by normalization completely.

The eigenstates are then

Case-1

$$\frac{b}{4} = \Re F(\eta) + aA\mathrm{tanh}\frac{\eta}{2}, \qquad g_2 = -ig_1 = -ig_d(\eta_1,\eta_2) \tag{E.17}$$

$$|V^{(\xi_1)}\rangle = g_d(\eta_1,\eta_2)\begin{pmatrix} 1 \\ -i \\ \sqrt{\frac{1}{b}}(\xi_1-1)\frac{1}{CU-\xi_1}|W\rangle \\ -i\sqrt{\frac{1}{b}}(\xi_1-1)\frac{1}{CU-\xi_1}|W\rangle \end{pmatrix} \tag{E.18}$$

$$|V^{(\bar{\xi}_2)}\rangle = g_d(\eta_2,\eta_1)\begin{pmatrix} 1 \\ -i \\ \sqrt{\frac{1}{b}}(\bar{\xi}_2-1)\frac{1}{CU-\bar{\xi}_2}|W\rangle \\ -i\sqrt{\frac{1}{b}}(\bar{\xi}_2-1)\frac{1}{CU-\bar{\xi}_2}|W\rangle \end{pmatrix} \tag{E.19}$$

Case-2

$$\frac{b}{4} = \Re F(\eta) - aA\mathrm{tanh}\frac{\eta}{2}, \qquad g_2 = ig_1 = ig_d(\eta_2,\eta_1) \tag{E.20}$$

$$|V^{(\xi_2)}\rangle = g_d(\eta_2,\eta_1)\begin{pmatrix} 1 \\ i \\ \sqrt{\frac{1}{b}}(\xi_2-1)\frac{1}{CU-\xi_2}|W\rangle \\ i\sqrt{\frac{1}{b}}(\xi_2-1)\frac{1}{CU-\xi_2}|W\rangle \end{pmatrix} \tag{E.21}$$

$$|V^{(\bar{\xi}_1)}\rangle = g_d(\eta_1,\eta_2)\begin{pmatrix} 1 \\ i \\ \sqrt{\frac{1}{b}}(\bar{\xi}_1-1)\frac{1}{CU-\bar{\xi}_1}|W\rangle \\ i\sqrt{\frac{1}{b}}(\bar{\xi}_1-1)\frac{1}{CU-\bar{\xi}_1}|W\rangle \end{pmatrix}. \tag{E.22}$$

Normalizing them in the following way[1]

$$\bar{V}^{\xi_i}_{\alpha}V^{\xi_j,\alpha} = \delta^{ij}$$

$$\bar{V}^{\bar{\xi}_i}_{\alpha}V^{\bar{\xi}_j,\alpha} = \delta^{ij}$$

$$\bar{V}^{\bar{\xi}}_{\alpha}V^{\xi,\alpha} = 0$$

we get, use the results of [50],

$$|g_d(\eta_1,\eta_2)|^2 = \frac{1}{2\Delta}\left[(1-r(\eta_1,\eta_2)) + r(\eta_1,\eta_2)\mathrm{sinh}\eta_1\frac{\partial}{\partial\eta_1}[\mathrm{Log}\Re F(\eta_1)]\right)^{-1}, \tag{E.23}$$

where

$$r(\eta_1,\eta_2) = \Re F(\eta_1)\frac{\tanh(\frac{\eta_1}{2})+\tanh(\frac{\eta_2}{2})}{\Re F(\eta_2)\tanh(\frac{\eta_1}{2})+\Re F(\eta_1)\tanh(\frac{\eta_2}{2})}. \tag{E.24}$$

It is important to notice that $V^{(\xi_1)}$ and $V^{(\bar{\xi}_1)}$ are degenerate eigenstates of $\mathcal{X}$, and the same holds for $V^{(\xi_2)}$ and $V^{(\bar{\xi}_2)}$.

E.2 Continuous spectrum

If ξ is in the continuous spectrum of CU ($\xi = \nu(k)$, [50]), we cannot invert the operator $(CU-\xi)$. Thus, in this case, the solution of (E.7) and (E.8) is

$$|\Lambda_1\rangle = A_1(k)|k\rangle + \frac{1}{\sqrt{b}}g_1(\nu(k)-1)\wp\frac{1}{(CU-\nu(k))}|W\rangle \tag{E.25}$$

$$|\Lambda_2\rangle = A_2(k)|k\rangle + \frac{1}{\sqrt{b}}g_2(\nu(k)-1)\wp\frac{1}{CU-\nu(k)}|W\rangle. \tag{E.26}$$

Where $\wp$ is the principal value, [50]. Using these in (E.5) and (E.6), we get

$$A_1(k) = g_1\sqrt{\frac{2}{b}}k\left(\Re F_c(k)-\frac{b}{4}\right) - \frac{\sqrt{2}Aa}{\sqrt{3b}}k\left(\frac{\nu(k)-1}{\nu(k)+1}\right)g_2 \tag{E.27}$$

$$A_2(k) = g_2\sqrt{\frac{2}{b}}k\left(\Re F_c(k)-\frac{b}{4}\right) + \frac{\sqrt{2}Aa}{\sqrt{3b}}k\left(\frac{\nu(k)-1}{\nu(k)+1}\right)g_1 \tag{E.28}$$

[1] This is the standard way to normalize eigenvectors of hermitian matrices

Note that in this case g_1 and g_2 are completely free and we can choose them to construct two linearly independent orthogonal vectors as follows

Case-1 $\quad g_2 = ig_1 = ig_c(k)$

$$V^1(k) = g_c(k) \left(\begin{array}{c} 1 \\ i \\ P(k)|k\rangle + \frac{1}{\sqrt{b}}(\nu(k)-1)\wp\frac{1}{CU-\nu(k)}|W\rangle - iH(k,a)|k\rangle \\ iP(k)|k\rangle + i\frac{1}{\sqrt{b}}(\nu(k)-1)\wp\frac{1}{CU-\nu(k)}|W\rangle + H(k,a)|k\rangle \end{array} \right) \tag{E.29}$$

Case-2 $\quad g_2 = -ig_1 = -ig_c(-k)$

$$V^2(k) = g_c(-k) \left(\begin{array}{c} 1 \\ -i \\ P(k)|k\rangle + \frac{1}{\sqrt{b}}(\nu(k)-1)\wp\frac{1}{CU-\nu(k)}|W\rangle + iH(k,a)|k\rangle \\ -iP(k)|k\rangle - i\frac{1}{\sqrt{b}}(\nu(k)-1)\wp\frac{1}{CU-\nu(k)}|W\rangle + H(k,a)|k\rangle \end{array} \right), \tag{E.30}$$

where

$$P(k) = \sqrt{\frac{2}{b}}k\left(\Re F_c(k) - \frac{b}{4}\right), \qquad H(k,a) = \frac{\sqrt{2}Aa}{\sqrt{3b}}k\left(\frac{\nu(k)-1}{\nu(k)+1}\right). \tag{E.31}$$

Imposing the continuous orthonormality condition

$$\bar{V}^{i,\alpha}(k)V^j_\alpha(k') = \delta^{ij}\delta(k-k') \tag{E.32}$$

we get

$$g_c(k) = \left[\frac{4\Delta}{b}N(k)\left(4 + k^2\left(\Re F_c(k) - \frac{b}{4} - \frac{Aa}{\tanh\frac{\pi k}{4}}\right)^2\right)\right]^{-1/2} \tag{E.33}$$

Sending $k \to -k$ we get the right degeneracy for $\mathcal{X}$.

E.3 Diagonalization of the 3-string vertex and the Lump state

We can express the oscillators $a^{(r)}_{N,\alpha}$, appearing in the 3-string vertex (7.72), in terms of the oscillators of the diagonal basis as

$$a^{(r)}_{N,\alpha} = \sum_{i=1}^{2}\left(a^{(r)}_{\xi_i}\bar{V}^{(\xi_i)}_{N,\alpha} + a^{(r)}_{\bar{\xi}_i}\bar{V}^{(\bar{\xi}_i)}_{N,\alpha} + \int_{-\infty}^{\infty} dk\, a^{(r)}_i(k)\bar{V}^{(i)}_{N,\alpha}(k)\right) \tag{E.34}$$

$$a^{(r)\dagger}_{N,\alpha} = \sum_{i=1}^{2}\left(a^{(r)\dagger}_{\xi_i}V^{(\xi_i)}_{N,\alpha} + a^{(r)\dagger}_{\bar{\xi}_i}V^{(\bar{\xi}_i)}_{N,\alpha} + \int_{-\infty}^{\infty} dk\, a^{(r)\dagger}_i(k)V^{(i)}_{N,\alpha}(k)\right). \tag{E.35}$$

Using these oscillators and the fact that $\tau\bar{V} = V$ ($\tau^\alpha_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$), we can rewrite the 3-string vertex as

$$|V_3^m\rangle = N_m \mathrm{exp}[-\frac{1}{2}\sum_{r,s}\sum_{i=1}^{2}\left(a_{\xi_i}^{(r)\dagger}\bar{V}_{N,\alpha}^{(\xi_i)} + a_{\bar{\xi}_i}^{(r)\dagger}\bar{V}_{N,\alpha}^{(\bar{\xi}_i)} + \int_{-\infty}^{\infty} dk a_i^{(r)\dagger}(k)\bar{V}_{N,\alpha}^{(i)}(k)\right)(\tau C'\mathcal{X})_{\beta,NM}^{\alpha(rs)}$$

$$\times\sum_{j=1}^{2}\left(a_{\xi_j}^{(s)\dagger}V_M^{(\xi_j)\beta} + a_{\bar{\xi}_j}^{(s)\dagger}V_M^{(\bar{\xi}_j)\beta} + \int_{-\infty}^{\infty} dk' a_j^{(s)\dagger}(k')V_M^{(j)\beta}(k')\right)]|\Omega_{b,\theta}\rangle \tag{E.36}$$

The twist operator $\tau C'$ acts on the eigenstates of the discrete and continous spectra as follows

$$\tau C' V^{(\xi_i)} = V^{(\bar{\xi}_i)} \qquad \tau C' V^{(i)}(k) = V^{(i+1)}(-k), \tag{E.37}$$

where $V^3(k)$ is identified with $V^1(k)$. Then (E.36) becomes

$$|V_3^m\rangle = N_m \mathrm{exp}\left[-\frac{1}{2}\sum_{r,s}\sum_{i=1}^{2}\left(a_{\xi_i}^{(r)\dagger}\mu^{rs}(\bar{\xi}_i)a_{\bar{\xi}_i}^{(s)\dagger} + a_{\bar{\xi}_i}^{(r)\dagger}\mu^{rs}(\xi_i)a_{\xi_i}^{(s)\dagger} + \right.\right.$$

$$\left.\left. + \int_{-\infty}^{\infty} dk a_i^{(r)\dagger}(k)\mu^{rs}(-k)a_{i+1}^{(s)\dagger}(-k)\right)\right]|\Omega_b\rangle. \tag{E.38}$$

In order to write this in an exact diagonal form, we need to introduce oscillators with definite τ–twist parity

$$e_{\xi_i}^r = \frac{a_{\xi_i}^r + a_{\bar{\xi}_i}^r}{\sqrt{2}} = \frac{a_{\xi_i}^r + \tau C' a_{\xi_i}^r}{\sqrt{2}}, \qquad o_{\xi_i}^r = -i\frac{a_{\xi_i}^r - a_{\bar{\xi}_i}^r}{\sqrt{2}} = -i\frac{a_{\xi_i}^r - \tau C' a_{\xi_i}^r}{\sqrt{2}} \tag{E.39}$$

$$e_i^r(k) = \frac{a_i^r(k) + a_{i+1}^r(-k)}{\sqrt{2}} = \frac{a_i^r(k) + \tau C' a_i^r(k)}{\sqrt{2}}$$

$$\tag{E.40}$$

$$o_i^r(k) = -i\frac{a_i^r(k) - a_{i+1}^r(-k)}{\sqrt{2}} = -i\frac{a_i^r(k) - \tau C' a_i^r(k)}{\sqrt{2}}$$

These oscillators have the following BPZ conjugation property

$$\mathrm{bpz}\ o_i = -o_i^\dagger \qquad \mathrm{bpz}\ e_i = -e_i^\dagger\ , \tag{E.41}$$

and satisfy the commutation relations

$$[e_{\xi_i}, e_{\xi_j}^\dagger] = \delta_{ij}, \qquad [o_{\xi_i}, o_{\xi_j}^\dagger] = \delta_{ij},$$

$$[e_i(k), e_j^\dagger(k')] = \delta_{ij}\delta(k-k'), \qquad [o_i(k), o_j^\dagger(k')] = \delta_{ij}\delta(k-k'), \tag{E.42}$$

with all the other commutators vanishing. Using them into (E.38) we finally obtain

$$|V_3^m\rangle = N_m \mathrm{exp}\left[-\frac{1}{4}\sum_{r,s}\sum_{i=1}^{2}\left(\left[\mu^{rs}(\xi_i) + \mu^{rs}(\bar{\xi}_i)\right]\left(e_{\xi_i}^{(r)\dagger}e_{\xi_i}^{(s)\dagger} + o_{\xi_i}^{(r)\dagger}o_{\xi_i}^{(s)\dagger}\right)\right.\right.$$

$$-i\left[\mu^{rs}(\xi_i) - \mu^{rs}(\bar{\xi}_i)\right]\left(o_{\xi_i}^{(r)\dagger}e_{\xi_i}^{(s)\dagger} - e_{\xi_i}^{(r)\dagger}o_{\xi_i}^{(s)\dagger}\right)$$

$$-\int_{-\infty}^{\infty} dk\mu^{rs}(k)\left(e_i^{(r)\dagger}(k)e_i^{(s)\dagger}(k)+o_i^{(r)\dagger}(k)o_i^{(s)\dagger}(k)\right)$$

$$-i\int_{-\infty}^{\infty} dk\mu^{rs}(k)\left(e_i^{(r)\dagger}(k)o_i^{(s)\dagger}(k)-o_i^{(r)\dagger}(k)e_i^{(s)\dagger}(k)\right)\Big)\Big]|\Omega_{b,\theta}\rangle \tag{E.43}$$

This gives the diagonal representation of the 3-string interaction vertex. The same procedure gives the following diagonal representation of the transverse part of the Lump

$$|S_\perp\rangle = \frac{A^2(3+4a^2)}{\sqrt{2\pi b^3}(\mathrm{Det}G)^{\frac{1}{4}}}\mathrm{Det}(\mathcal{I}-\mathcal{X})^{\frac{1}{2}}\mathrm{Det}(\mathcal{I}+\mathcal{T})^{\frac{1}{2}}\exp\left(-\frac{1}{2}\sum_{i=1}^{2}\Big[t_d(\eta_i)\left(e^\dagger_{\xi_i}e^\dagger_{\xi_i}+o^\dagger_{\xi_i}o^\dagger_{\xi_i}\right)+\right.$$

$$\left.+\frac{1}{2}\int_{-\infty}^{\infty} dkt_c(k)\left(e_i^\dagger(k)e_i^\dagger(k)+o_i^\dagger(k)o_i^\dagger(k)\right)\Big]\right)|\Omega_{b,\theta}\rangle \tag{E.44}$$

E.4 Asymptotic behaviors

In Section 3.3, we have analyzed our solution in the large and small limits of the parameter b. In this appendix we compute the relevant matrix elements in these asymptotic regimes.

E.4.1 The $b \to \infty$ Limit

From (E.16), we can write

$$|a| = \frac{\Re F(\eta_2)-\Re F(\eta_1)}{[V_{00}+2\Re F(\eta_2)]\tanh(\frac{\eta_1}{2})+[V_{00}+2\Re F(\eta_1)]\tanh(\frac{\eta_2}{2})} \tag{E.45}$$

$$\frac{b}{4} = \frac{\Re F(\eta_2)\tanh(\frac{\eta_1}{2})+\Re F(\eta_1)\tanh(\frac{\eta_2}{2})}{\tanh(\frac{\eta_1}{2})+\tanh(\frac{\eta_2}{2})} \tag{E.46}$$

where we take, by definition, $\eta_2 > \eta_1 > 0$. There are two ways of taking $b \to \infty$

i) $\eta_2 \to \infty$; η_1 fixed

In this limit we can see that

$$\frac{b}{4} \approx \left(\frac{\tanh(\frac{\eta_1}{2})}{1+\tanh(\frac{\eta_1}{2})}\right)\log(\eta_2), \qquad a \approx \frac{1}{2\tanh(\frac{\eta_1}{2})} > \frac{1}{2}. \tag{E.47}$$

ii) $\eta_2 \to \infty$; $\eta_1 \to \infty$

We can parameterize $\eta_2 = \eta^y$, $\eta_1 = \eta^x$ and then take $\eta \to \infty$, while keeping $y > x$. We then obtain

$$\frac{b}{4} \approx \frac{1}{2}(y+x)\log(\eta), \qquad a \approx \frac{1}{2}\frac{y-x}{y+x} < \frac{1}{2} \tag{E.48}$$

We will be concerned with this second regime as it is the one connected to $a = 0$, which is a condition arising from the existence of the critical value for the E–field, when $b \to \infty$. In this second limit it can be easily seen that the discrete eigenvectors have the following behaviour

$$V_0^{\xi_i,1} = V_0^{\bar{\xi}_i,1} \approx \frac{1}{\sqrt{2\Delta}}e^{-\eta_i/2}\sqrt{\eta_i\mathrm{Log}\eta_1\eta_2},$$

$$V_0^{\xi_i,2} = -V_0^{\bar{\xi}_i,2} \approx (-1)^i \frac{i}{\sqrt{2\Delta}} e^{-\eta_i/2} \sqrt{\eta_i \mathrm{Log}\eta_1\eta_2},$$

and

$$V_n^{\xi_i,\alpha} \approx -\frac{V_0^{\xi_i,\alpha}}{\sqrt{\mathrm{Log}\eta_1\eta_2}}, \qquad V_n^{\bar{\xi}_i,\alpha} \approx -\frac{V_0^{\bar{\xi}_i,\alpha}}{\sqrt{\mathrm{Log}\eta_1\eta_2}}. \tag{E.49}$$

For the continuous spectrum the situation is more complicated and getting this limits is not easy. However, it is possible to calculate the limit of $(V_0^{i,\alpha}(k))^2$, which is enough for our purposes. We have

$$(V_0^{1,1}(k))^2 = \left[\frac{4\Delta}{b} N(k) \left(4 + k^2 \left(\Re F_c(k) - \frac{b}{4} - \frac{Aa}{\tanh\frac{\pi k}{4}}\right)^2\right)\right]^{-1}. \tag{E.50}$$

When $b \to \infty$ this expression vanishes every where except at

$$k_0 \approx -\frac{4}{\pi} \mathrm{arctanh}(2a) \tag{E.51}$$

where it diverges. Expanding around k_0 one easily gets

$$(V_0^{1,1}(k))^2 \approx \Delta^{-1} \frac{4a}{k_0(1-4a^2)N(k_0)} \frac{\bar{b}}{\pi(1+(k-k_0)^2\bar{b}^2}$$

where

$$\bar{b} = \frac{k_0\pi(1-4a^2)}{64a} b.$$

Now taking the $b \to \infty$ limit one obtains

$$(V_0^{1,1}(k))^2 \approx \frac{1}{2\Delta} \delta(k-k_0). \tag{E.52}$$

Following the same procedure one can also show that

$$(V_0^{2,1}(k))^2 \approx \frac{1}{2\Delta} \delta(k+k_0) \tag{E.53}$$

remember that

$$|V_0^{1,2}(k)|^2 = (V_0^{1,1}(k))^2, \qquad |V_0^{2,2}(k)|^2 = (V_0^{2,1}(k))^2. \tag{E.54}$$

The non zero components, $V_m^{i,\alpha}(k)$, can be expressed in terms of a generating function. For instance, the generating function for $V_m^{1,1}(k)$ is given by

$$F^{(k)}(z) = A_1(k) f^{(k)}(z) - \frac{(1-\nu(k))V_0^{1,1}(k)}{\sqrt{b}} B(k,z) \tag{E.55}$$

where

$$A_1(k) = V_0^{1,1}(k) \sqrt{\frac{2}{b}} k \left(\Re F_c(k) - \frac{b}{4} - \frac{Aa}{\tanh(\frac{\pi k}{4})}\right),$$

$$B(k,z) = \frac{2}{1-\nu(k)} \left[\Re F_c(k) + \frac{\pi}{2\sqrt{3}} \frac{\nu(k)-1}{\nu(k)+1} + \frac{2i}{k} + \log(iz) - 2if^{(k)}(z)\right]$$

$$+\frac{2}{1-\nu(k)}\left[\Phi(e^{-4i\arctan(z)},1,1+\frac{k}{4i})e^{-4i\arctan(z)}e^{-k\arctan(z)}\right] \tag{E.56}$$

where Φ is the LerchPhi function and $f^{(k)}$ is the generating function for the spectrum of the Neumann matrix without zero modes, [28]. Inverting this equation we can write $V_m^{1,1}(k)$ as

$$V_m^{1,1}(k)=A_1(k)\frac{\sqrt{m}}{2\pi i}\oint dz\frac{f^{(k)}(z)}{z^{m+1}}-\frac{(1-\nu(k))V_0^{1,1}(k)}{\sqrt{b}}\frac{\sqrt{m}}{2\pi i}\oint dz\frac{B(k,z)}{z^{m+1}} \tag{E.57}$$

With the same procedure one can also write

$$V_m^{2,1}(k)=A_1'(k)\frac{\sqrt{m}}{2\pi i}\oint dz\frac{f^{(k)}(z)}{z^{m+1}}-\frac{(1-\nu(k))V_0^{2,1}(k)}{\sqrt{b}}\frac{\sqrt{m}}{2\pi i}\oint dz\frac{B(k,z)}{z^{m+1}} \tag{E.58}$$

with

$$A_1'(k)=V_0^{2,1}(k)\sqrt{\frac{2}{b}}k\left(\Re F_c(k)-\frac{b}{4}+\frac{Aa}{\tanh(\frac{\pi k}{4})}\right). \tag{E.59}$$

The other vectors are related to these ones as

$$V_n^{1,2}(k)=iV_n^{1,1}(k), \qquad V_n^{2,2}(k)=-iV_n^{2,1}(k) \tag{E.60}$$

E.4.2 Limit of $\hat{S}_{mn}^{\alpha\beta(c)}$

With all these results at hand we can now calculate the continuous spectrum contribution to the non zero mode matrix elements in the limits under consideration. Recalling that spectrum of the Neumann matrix without zero modes is given by

$$v_m^{(k)}=\frac{\sqrt{m}}{2\pi i}\oint dz\frac{f^{(k)}(z)}{z^{m+1}} \tag{E.61}$$

we can write

$$\hat{S}_{nm}^{11(c)}=\int_{-\infty}^{\infty}dk\ t_c(k)(-1)^n\left[V_n^{1,1}(k)\bar{V}_m^{1,1}(k)+V_n^{2,1}(k)\bar{V}_m^{2,1}(k)\right] \tag{E.62}$$

as

$$\begin{aligned}\hat{S}_{mn}^{11(c)}=&\int_{-\infty}^{\infty}dk\ t_c(k)(-1)^m[A_1(k)A_1(k)v_m^{(k)}v_n^{(k)}-A_1(k)V_0^{1,1}(k)v_m^{(k)}(1-\bar{\nu}(k))\tilde{B}_n(k)\frac{1}{\sqrt{b}}\\&-A_1(k)V_0^{1,1}(k)v_n^{(k)}(1-\nu(k))\tilde{B}_m(k)\frac{1}{\sqrt{b}}+(V_0^{1,1}(k))^2(1-\bar{\nu}(k))(1-\nu(k))\tilde{B}_m(k)\tilde{B}_n(k)\frac{1}{b}]\\&+\left[A_1(k)\to A_1'(k),V_0^{1,1}(k)\to V_0^{2,1}(k)\right]\end{aligned} \tag{E.63}$$

where

$$\tilde{B}_m(k)=\frac{\sqrt{m}}{2\pi i}\oint dz\frac{B(k,z)}{z^{m+1}}. \tag{E.64}$$

Note that if the indices are separated by comma then the first index is the label of the vector and the second is the space time index, otherwise both are space time indices. Now we want to calculate each term in the above expression in the limit when $b \to \infty$. To this end we notice the following

$$\lim_{b\to\infty} A_1(k)A_1(k) = \lim_{b\to\infty} (V_0^{1,1}(k))^2 \left(\frac{2k^2}{b}\right)\left(\Re F_c(k) - \frac{b}{4} - \frac{Aa}{\tanh(\frac{\pi k}{4})}\right)^2$$

$$= \lim_{x\to-\infty} \left(\frac{k^2}{2\Delta N(k)}\right) \frac{x^2}{4+k^2x^2} = \left(\frac{k^2}{2\Delta N(k)}\right)\frac{1}{k^2} = \frac{1}{2\Delta N(k)} \tag{E.65}$$

where $x = \left(\Re F_c(k) - \frac{b}{4} - \frac{Aa}{\tanh(\frac{\pi k}{4})}\right)$. The other terms are zero in the limit because, either they contain term like $(k-k_0)\delta(k-k_0)$ in the integral or they are of order $\frac{1}{b}$. Therefore, we are left with

$$\lim_{b\to\infty} \hat{S}^{11(c)}_{mn} = \lim_{b\to\infty} \hat{S}^{22(c)}_{mn} = \Delta^{-1} S_{mn}, \quad \text{where} \quad S_{nm} = -\int_{-\infty}^{\infty} \frac{dk\ t_c(k)}{N(k)} v_n^{(k)} v_m^{(-k)} \tag{E.66}$$

and

$$\lim_{b\to\infty} \hat{S}^{21(c)}_{mn} = \lim_{b\to\infty} \hat{S}^{12(c)}_{mn} = 0, \tag{E.67}$$

which is the sliver in each direction with corrections of order $\frac{1}{b}$.

E.4.3 Limit of $\hat{S}^{\alpha\beta(c)}_{0m}$

In this section we would like to justify that the contribution from the continuous spectrum to $\hat{S}^{\alpha\beta}_{0m}$ is zero in the limit. This can be computed the same way as before since we have

$$\lim_{b\to\infty} \hat{S}^{\alpha\beta(c)}_{0m} = \lim_{b\to\infty} \sum_{i=1}^{2} \int_{-\infty}^{\infty} dk\ t_c(k) V_0^{i,\alpha}(k) V_m^{i,\beta}(k). \tag{E.68}$$

For instance, lets calculate $\hat{S}^{11(c)}_{0m}$ which is given by

$$\hat{S}^{11(c)}_{0m} = \lim_{b\to\infty} \int_{-\infty}^{\infty} dk\ t_c(k) v_m^{(k)} \sqrt{\frac{2}{b}} k \left((V_0^{1,1}(k))^2 \left[\Re F_c(k) - \frac{b}{4} - \frac{Aa}{\tanh(\frac{\pi k}{4})}\right] \right.$$

$$\left. + (V_0^{2,1}(k))^2 \left[\Re F_c(k) - \frac{b}{4} + \frac{Aa}{\tanh(\frac{\pi k}{4})}\right]\right) + O(\frac{1}{\sqrt{b}}) \tag{E.69}$$

We have already verified that $\lim_{b\to\infty}(V_0^{i,\alpha}(k))^2 \approx \frac{1}{2}\delta(k \pm k_0)$. This will allow us to expand the terms in square brackets about the points $\pm k_0$ to get

$$\hat{S}^{11(c)}_{0m} = \frac{1}{\Delta} \lim_{b\to\infty} \int_{-\infty}^{\infty} dk\ t_c(k) v_m^{(k)} \left[\frac{1}{2}\delta(k-k_0)\sqrt{b}(k-k_0)k_0\pi\left(\frac{1-4a^2}{32a}\right)\right.$$

$$\left. + \frac{1}{2}\delta(k+k_0)\sqrt{b}(k+k_0)(-k_0)\pi\left(\frac{1-4a^2}{32a}\right)\right] + O(\frac{1}{\sqrt{b}}). \tag{E.70}$$

Due to the presence of the delta functions the terms $(k \pm k_0)\sqrt{b}$ are both finite in the $b \to \infty$ limit. As a matter of this fact we can safely do the integrals first and take the limits later. Since the integrals vanishes we note that

$$\hat{S}_{0m}^{11(c)} \approx 0. \tag{E.71}$$

Similar steps show that all the remaining terms of $\hat{S}_{0m}^{\alpha\beta(c)}$ are also zero.

E.4.4 The $b \to 0$ Limit

As it was mentioned before this limit can be obtained by taking $\eta_1 \to 0$. In this limit it is not hard to see that

$$b \approx 2\frac{\Re F(\eta_2)}{\tanh(\frac{\eta_2}{2})}\eta_1 \tag{E.72}$$

$$g_d(\eta_1, \eta_2) \approx \frac{1}{\sqrt{2\Delta}}\left(1 - \frac{\tanh(\frac{\eta_2}{2})}{2\Re F(\eta_2)}\eta_1\right) \tag{E.73}$$

$$g_d(\eta_2, \eta_2) \approx \frac{1}{\sqrt{2\Delta}}\left[2\tanh(\frac{\eta_2}{2})\left(\sinh\eta_2\frac{\partial}{\partial\eta_2}[\mathrm{Log}\Re F(\eta_2)] - 1\right)\right]^{-1/2}\sqrt{\eta_1}. \tag{E.74}$$

One can use these results and equations (E.18) through (E.22) to write down $V_0^{\xi_i,\alpha}$, $V_0^{\bar{\xi}_i,\alpha}$ $V_n^{\xi_i,\alpha}$ and $V_n^{\bar{\xi}_i,\alpha}$ as

$$\begin{aligned} V_0^{\xi_1,1} = V_0^{\bar{\xi}_1,1} &\approx \frac{1}{\sqrt{2\Delta}}\left(1 - \frac{\tanh(\frac{\eta_2}{2})}{2\Re F(\eta_2)}\eta_1\right), \\ V_0^{\xi_1,2} = -V_0^{\bar{\xi}_1,2} &\approx -i\frac{1}{\sqrt{2\Delta}}\left(1 - \frac{\tanh(\frac{\eta_2}{2})}{2\Re F(\eta_2)}\eta_1\right), \end{aligned} \tag{E.75}$$

$$\begin{aligned} V_0^{\xi_2,1} = V_0^{\bar{\xi}_2,1} &\approx \frac{1}{\sqrt{2\Delta}}\left[2\tanh(\frac{\eta_2}{2})\left(\sinh\eta_2\frac{\partial}{\partial\eta_2}[\mathrm{Log}\Re F(\eta_2)] - 1\right)\right]^{-1/2}\sqrt{\eta_1} \\ V_0^{\xi_2,2} = -V_0^{\bar{\xi}_2,2} &\approx i\frac{1}{\sqrt{2\Delta}}\left[2\tanh(\frac{\eta_2}{2})\left(\sinh\eta_2\frac{\partial}{\partial\eta_2}[\mathrm{Log}\Re F(\eta_2)] - 1\right)\right]^{-1/2}\sqrt{\eta_1}, \end{aligned} \tag{E.76}$$

and

$$\begin{aligned} V_n^{\xi_1,\alpha} = \pm V_n^{\bar{\xi}_1,\alpha} &\approx \sqrt{\eta_1}, \\ V_n^{\xi_2,\alpha} = \pm V_n^{\bar{\xi}_2,\alpha} &\approx \frac{1}{\sqrt{2\Delta}}\left[2\tanh(\frac{\eta_2}{2})\left(\sinh\eta_2\frac{\partial}{\partial\eta_2}[\mathrm{Log}\Re F(\eta_2)] - 1\right)\right]^{-1/2} f(\eta_2). \end{aligned} \tag{E.77}$$

The f is a regular function of η_2. On the other hand

$$g_c(k) \approx 0. \tag{E.78}$$

This shows all $V_0^{i,\alpha}(k)$ are zero, whereas $V_m^{i,\alpha}(k)$ are finite and b independent to the leading order. These results are extensively used in section 3.3 to calculate quantities like s_1, s_2 in the $b \to 0$ limit.

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